Dynamic Pricing, Scheduling, and Order Rejection of a Multiclass Omnichannel Hybrid Production System

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Abstract. Many production systems now fulfill demand through both walk-in and online channels, producing some goods to order and others to stock on shared capacity. Coordinating production, pricing, and admission across these channels and fulfillment modes is a central operational challenge in these systems. Motivated by such settings, we study joint dynamic pricing, scheduling, and order rejection in a multiclass omnichannel hybrid production system in which the firm offers make-to-order and make-to-stock goods through walk-in and online channels, with online customers selecting from multiple quote times for future pickup. Make-to-order goods incur earliness and tardiness costs, while make-to-stock goods incur holding and tardiness costs. Walk-in customers may abandon if their waiting time is excessive. We model this problem as a stochastic processing network and analyze it in the heavy-traffic regime, approximating the original control problem by a Brownian control problem. We then reduce it to an equivalent workload formulation and solve it in closed form. The optimal policy depends on the aggregate congestion in the system, with the workload space partitioned into three regions across which the structure of the policy changes. At moderate congestion, the firm exploits a scheduling buffer created by the quote times of the online classes, incurring no congestion cost. At low congestion, this scheduling buffer cannot be fully utilized, so earliness or holding costs are unavoidable, and the firm idles deliberately when these costs become excessive. At high congestion, the scheduling buffer is insufficient, so tardiness or abandonment costs are unavoidable, and the firm rejects orders when these costs become excessive. Pricing regulates congestion by adjusting demand in response to the system workload. Building on this structure, we propose three variants of a dynamic control policy that differ in their degree of pricing flexibility: fully dynamic, online-only dynamic, and static pricing. In a numerical study, we demonstrate that the proposed policy is effective and that both dynamic pricing and congestion-aware static pricing deliver significant gains. Moreover, dynamic pricing delivers the largest gains when demand is concentrated online or in make-to-stock goods. Notably, online-only dynamic pricing captures most of these gains, providing a practical entry point for firms seeking to implement dynamic pricing.

Key words: omnichannel, pricing, scheduling, make-to-order, make-to-stock, heavy traffic analysis, stochastic control

1. Introduction

Omnichannel systems, which integrate multiple demand channels, have become increasingly prevalent across many industries, such as hospitality, healthcare, and retail. Production systems have also adopted omnichannel practices, giving rise to omnichannel production, in which production and fulfillment are dynamically coordinated across multiple channels. These systems arise in a variety of settings, including fast-casual restaurants, coffee chains, pharmacies, and specialty production (e.g., florists), where walk-in and online orders compete for shared fulfillment capacity. Major foodservice chains, including Starbucks, Panera Bread, Sweetgreen, and Chipotle, provide a salient example of this trend. In such settings, the scale of digital demand is already substantial: digital orders exceed 30% of sales at Starbucks and Chipotle and 50% at Panera Bread and Sweetgreen.[1] These figures underscore the growing operational importance of effectively managing the online channel alongside in-person demand.

While the integration of multiple ordering channels enhances system flexibility and expands market reach, it also creates operational complexities that firms are increasingly addressing through automated order management systems rather than manual decision making. In particular, firms must coordinate fulfillment across both walk-in and online channels while managing the production of make-to-order (MTO) and make-to-stock (MTS) goods. For example, in foodservice chains, customizable meals and beverages are produced only after an order is received, whereas uncustomized premade items, such as some house-made cold drinks, salads, sandwiches, and grab-and-go items, are prepared in advance and stocked for immediate sale. In pharmacies, common high-demand medications may be prepared in advance, whereas specialty medications and compounded prescriptions often require preparation only after an order is received. In floral retail stores, standard bouquets and display arrangements may be prepared ahead of time, whereas customized arrangements are assembled only after demand is realized and must often be completed by a promised pickup time. These two production modes generate fundamentally different costs. MTO goods require careful timing, especially for online orders: if they are prepared too early, they may incur earliness costs as quality can deteriorate before the customer arrives—leading, for example, to "lukewarm lattes" or "cold burritos" (Ghosh et al. 2026). In foodservice chains, this issue is operationally important: a recent industry report finds that 20% of customers cite receiving cold food as their biggest frustration (TouchBistro 2024). Conversely, MTS goods can be prepared in advance and incur holding costs when held in inventory. Regardless of the type of good, tardiness costs arise when orders are delayed. Further complicating matters, walk-in customers in particular may abandon their purchase entirely if waiting times become too long.

These trade-offs make scheduling a critical operational challenge in omnichannel hybrid production systems, and one that the existing literature has not adequately addressed. In particular, firms must decide not only what to produce, but also when to produce it and how to sequence production across products and channels. The challenge is widely recognized in practice. For example, Starbucks CEO Brian Niccol stated in a recent interview that the company's mobile ordering system often quotes pickup times that are mismatched with customer arrival times, leading to orders being prepared too early. He further emphasized that customers' number one request is the ability to select their pickup time, underscoring the importance of offering multiple quote times and dynamically coordinating fulfillment across multiple goods, channels, and quote times (Haddon 2025, Bitter 2025). Consistent with this concern, the *WSJ* reports that Starbucks is transitioning from its first-come, first-served policy to a rules-based sequencing policy, which has reduced waiting times at pilot locations (Haddon and Bousquette 2025).

Beyond scheduling, firms are increasingly exploring demand-side controls, such as pricing, to manage congestion. Pricing in such systems has traditionally been guided by deterministic models that rely only on expected demand and capacity. Yet prices do more than balance expected demand and capacity: they shape the arrival process, and with it, waiting times and congestion costs. Accounting for congestion is therefore essential, even when static prices are used.

In the foodservice industry, the growing adoption of digital menus and online ordering has expanded the scope for demand-side control, enabling prices to be adjusted dynamically in response to operating conditions. While congestion-based pricing is relatively new to this industry, several startups, including Sauce and Priceff, have introduced dynamic pricing tools for fast-casual restaurants that adjust prices based on operating conditions. These tools have already shown promise in initial rollouts at chains such as Dog Haus and Kotipizza. Initial evidence suggests that modest price adjustments can ease the load on the kitchen and reduce waiting times (Hynum 2021, Kaiser 2022, Webb 2023). Dynamic pricing is thus emerging as an important operational lever for managing congestion in omnichannel hybrid production systems. Despite this, the existing literature has not addressed pricing in omnichannel or hybrid production systems.

Motivated by these operational challenges and recent developments in practice, we study the dynamic control of omnichannel production systems, particularly settings in which key operational decisions are increasingly automated. To that end, we develop a stochastic processing network model for a multiclass omnichannel hybrid production system that serves heterogeneous, price- and delay-sensitive customers. The firm offers both MTO and MTS goods through walk-in and online channels, with online customers selecting from multiple predetermined quote times for future pickup. MTO orders incur earliness and tardiness costs, while MTS orders incur tardiness and holding costs. Walk-in customers are impatient and may abandon the system if waiting times are excessive, resulting in abandonment costs. The firm seeks to maximize its long-run average expected profit by jointly making pricing, scheduling, and order rejection decisions, under one of three degrees of pricing flexibility: (fully) dynamic, online-only dynamic, or static. Scheduling decisions consist of two types of activities: production and reallocation. Production activities involve manufacturing goods, while reallocation activities redirect available MTS inventory to satisfy online orders. Reallocation activities are needed because, unlike MTO goods, which must be produced specifically for each order, MTS goods are inherently flexible in that they can be used to satisfy demand from any channel or quote time.

Because the joint pricing, scheduling, and rejection control problem is analytically and computationally intractable, we adopt the methodology proposed by Harrison (1988, 2003) and analyze the system in heavy traffic. We approximate the original control problem by a Brownian control problem, and prove that it is equivalent to a one-dimensional drift rate control problem, whose workload process represents the total backlog in the system measured in hours of work. We solve this drift rate control problem by solving the associated Bellman equation and establish that the optimal policy is a two-sided barrier policy with a state-dependent drift rate, for which we derive a closed-form solution. The idling and rejection processes maintain the workload process between the lower and upper barriers, respectively. Between the two barriers, the drift rate and workload configuration policies control the workload.

We translate the solution to the workload formulation into an easily implementable control policy for the original production system. This policy steers the workload distribution toward the optimal workload configuration by prioritizing classes according to their effective state costs, which capture both immediate congestion costs and forward-looking components that capture the impact of abandonment. Under dynamic pricing, our policy dynamically regulates the demand rate, whereas under static pricing, a fixed price vector regulates the demand rate in anticipation of congestion. Rejection (resp., idling) is triggered only when the workload reaches the upper (resp., lower) barrier.

As discussed earlier, scheduling and pricing are central decisions in omnichannel hybrid production systems, yet neither has been studied in this setting. We propose a joint pricing, scheduling, and rejection policy with a simple and interpretable structure and make the following contributions:

Modeling Contributions. We develop a stochastic processing network model for a multiclass omnichannel hybrid production system in which the firm jointly controls pricing, scheduling, and rejection. The model provides a unified treatment of multiple ordering channels, multiple quote times, hybrid production, and abandonment within a single dynamic control problem. The joint presence of these features generates fundamentally new operational and cost trade-offs that do not arise in single-channel or non-hybrid settings.

A central contribution of the model is a cost structure tailored to omnichannel hybrid production. MTO goods must be synchronized with promised pickup times and thus generate earliness and tardiness costs, while MTS goods can be produced in advance, generating holding and tardiness costs while also providing operational flexibility. MTS inventory is fungible across channels and quote times; it can be redirected to serve demand in either channel or at any quote time, requiring a stochastic processing network framework.

Technical Contributions. This paper makes several technical contributions. First, we solve a drift rate control problem with two novel structural features: (i) a piecewise-linear state cost function with multiple breakpoints induced by the interaction between the online channel and hybrid production; and (ii) the state cost cannot be decoupled from the value function, since customer abandonment induces a state-dependent drift rate that links the state cost to the derivative of the value function, rendering the optimal workload configuration nongreedy and the breakpoints endogenous. Second, despite the complexity of the control problem, we prove that the Bellman equation admits a closed-form solution. We construct the solution by solving a sequence of initial value problems over subintervals induced by the piecewise-linear cost structure and smoothly pasting them together. Third, we prove that the optimal policy is a two-sided barrier policy with state-dependent drift rate and workload configuration, and characterize it in closed form.

Structural Insights. The technical results yield an intuitive characterization of how congestion should be managed in a hybrid omnichannel production system. In particular, the optimal solution to the workload formulation partitions the workload space into three regions, determined by two thresholds induced by the quote times of the online classes. Each region corresponds to a distinct mechanism for absorbing work. In the intermediate-congestion region, the online channel plays a central role. Online products, by virtue of their positive quote times, create a scheduling buffer: orders placed now need not be completed until the customer arrives, so the firm can hold workload in these classes without incurring state costs.

In the low-congestion region, the scheduling buffer cannot be fully utilized, as the workload is insufficient to align production completion times with quoted pickup times without idling or incurring holding costs. Some form of inefficiency is therefore unavoidable: the firm must either produce MTO goods ahead of schedule, incurring earliness costs, or carry excess MTS inventory, incurring holding costs. The optimal policy chooses the cheapest combination of these two sources of cost. When the workload reaches the lower barrier, the server idles deliberately rather than accumulating additional earliness or holding costs.

In the high-congestion region, the scheduling buffer is fully utilized, and excess workload unavoidably incurs tardiness or abandonment costs. The optimal policy allocates this excess workload to the class with the lowest aggregate tardiness and abandonment cost per unit of work, which may depend on the workload level. This dependence arises because the scheduling policy must balance the immediate state costs against the impact of abandonment on future congestion. The class that is cheapest at one congestion level may not remain cheapest at higher levels, making the policy nongreedy.

Managerial Implications. The numerical study demonstrates that the proposed policies with static and dynamic prices closely track their respective semi-Markov decision process (SMDP) benchmarks in performance, pricing, and workload distributions, indicating that the aggregate workload captures most of the economically relevant state information. The computational advantage is substantial: even in a simple four-product setting, the workload formulation can be solved in closed form in seconds, whereas the SMDP benchmark requires days. The performance gains are also substantial. In the base case, the proposed policy with static prices improves performance by over 70% relative to a benchmark that fixes prices at the values prescribed by a deterministic model that ignores congestion and stochastic variability. For firms with dynamic pricing capability, dynamic pricing delivers a further 27% improvement over static pricing.

The numerical study yields several broader managerial insights. First, demand composition is a key determinant of the value of dynamic pricing. As demand shifts toward the online channel or toward MTS goods, the value of dynamic pricing increases. The mechanism is intuitive: online products provide a scheduling buffer that absorbs congestion without incurring state costs, and MTS goods can be produced in advance and reallocated as needed. As a result, the firm visits the costly high-congestion region less frequently. Dynamic pricing is most valuable precisely in this setting because it can selectively raise prices during infrequent congestion spikes, an ability that static pricing lacks. Therefore, firms with substantial online demand, or the ability to shift demand online, stand to benefit most from dynamic pricing.

The results outline a practical adoption path for implementing scheduling and pricing. Firms can begin with the proposed policy with static prices if their primary challenge is scheduling. They can then adopt the proposed policy with online-only dynamic prices as an intermediate step, and transition to the proposed policy with dynamic prices as their capabilities mature. Specifically, when a substantial share of demand is online, online-only dynamic pricing already captures most of the benefits of full dynamic pricing, making it a natural entry point for firms seeking to implement dynamic pricing gradually.

Second, the proposed policy is driven primarily by the aggregate workload, which evolves on a slower time scale than the underlying arrival and production processes. The policy therefore responds to coarse changes in congestion rather than to instantaneous fluctuations in individual queue lengths. Dynamic pricing also does not require large price swings: prices remain within 13% of the nominal static prices across all policies and products, consistent with regulatory preferences that discourage large price changes.

Third, abandonment partially substitutes for dynamic pricing by relieving congestion, but it does so at high cost, whereas dynamic pricing shapes demand optimally. Furthermore, when abandonment rates are moderate, the gains from dynamic pricing remain substantial. Finally, dynamic pricing delivers the highest gains in heavily loaded systems and in settings where products are not close substitutes.

Organization. The rest of this paper is organized as follows. Section 2 reviews the literature. Section 3 presents the model and control problem. Section 4 introduces a Brownian approximation of the original control problem, which Section 5 reduces to an equivalent workload formulation. Section 6 solves the workload formulation, and Section 7 interprets its solution in the context of the original control problem and proposes three policy variants with different degrees of price flexibility. Section 8 examines the structure of our proposed policy, evaluates its effectiveness through an extensive numerical study, and offers managerial insights. The appendices supplement the main body by formally deriving the Brownian control problem (Appendix EC.1), solving the Bellman equation (Appendix EC.2), proving the theoretical results (Appendices EC.3 and EC.4), and providing supplementary material for the numerical study (Appendix EC.5).

2. Literature Review

Most work on omnichannel operations focuses on retail settings, where common research questions include inventory allocation for online orders (Gao et al. 2023b, Baron et al. 2024), return policies (Nageswaran et al. 2020), and information sharing (Roet-Green and Yang 2024); see Hübner et al. (2022) for an overview. Omnichannel retail and omnichannel production, however, pose distinct operational challenges. In retail, demand from multiple channels is fulfilled from existing inventory, and the central question is how to allocate that inventory across locations and orders. In production, every order must be produced in-house using shared production capacity, and the online channel affects not only the volume and composition of demand but also the timing of production due to the presence of quote times. The central question is therefore the dynamic allocation of capacity over time. Closest to our setting is the small body of work on omnichannel service systems, where demand, as in production, must be met through server capacity rather than existing inventory. For example, Kang et al. (2024) study strategic joining and prioritization in a two-class system. The closest and most notable contribution in this literature is Gao et al. (2023a), which studies dynamic scheduling and order rejection in a two-class omnichannel service system and is discussed further below.

The omnichannel nature of our setting, particularly the presence of an online channel with quote times, connects our work to the literature on lead time quotation, where the central issue is how firms should make and honor promised completion times. Much of this literature treats due dates as hard constraints (Plambeck 2004, Çelik and Maglaras 2008, Huang et al. 2015). Our setting, by contrast, allows orders to be completed either before or after the quote time, generating earliness and tardiness costs. A smaller body of work considers this by modeling the cost of completing work too early or too late. For example, Farahani et al. (2022) consider an online food-ordering system with a single MTO good offered at a single quote time, and show that deliberate idling can be valuable when early production is costly. The closest related work to ours in this respect is Gao et al. (2023a), which studies dynamic scheduling and order rejection in a service system with a single MTO good offered through the walk-in channel and through the online channel at a single quote time. In our paper, the system offers multiple goods, the online channel quotes multiple pickup times, MTS and MTO goods share production capacity, and customers may abandon. Moreover, in addition to scheduling and rejection, the system manager uses pricing to control the system. These features lead to a fundamentally different control problem with novel trade-offs and new managerial insights.

The literature on the dynamic control of production systems with a single demand channel is vast, with most work examining either an MTO or an MTS system; see Nahmias and Olsen (2015, Chapter 9) for an overview. MTO production has been extensively studied, with work examining lead time quotation (Plambeck 2004, Ata and Olsen 2009, Afèche and Pavlin 2016), admission control (Ata 2006), scheduling (Rubino and Ata 2009, Sun and Zhu 2025), and pricing (Çelik and Maglaras 2008). The MTS stream has received considerably less attention. One of the earliest contributions is Wein (1992), which proposes a dynamic scheduling policy for a multiclass MTS system using heavy-traffic analysis. This framework has since been extended to index-based scheduling (Veatch and Wein 1996, Perez and Zipkin 1997) and, more recently, to dynamic pricing (Ata and Barjesteh 2022). A related strand of work studies assemble-to-order systems, in which components are produced in advance and assembled to order (Plambeck and Ward 2006, Nadar et al. 2018, DeValve et al. 2020).

Despite their prevalence in practice, hybrid production systems, which produce both MTO and MTS goods, have received little attention; see Peeters and van Ooijen (2020) for an overview. An early paper is Carr and Duenyas (2000), which studies joint scheduling and admission control for a two-product hybrid production system where MTS demand must be met through available inventory or an external supplier at a cost, while MTO demand can be rejected. Iravani et al. (2012) extend this setting by allowing backlogging for MTS orders. The closest paper to ours is Markowitz and Wein (2001), which studies the scheduling of a multiclass hybrid production system with switching costs in heavy traffic. Due to switching costs, they focus on dynamic cyclic policies, which they analyze computationally. The control problem we study has a different economic structure: the absence of switching costs permits a broader class of scheduling policies; at the same time, our paper considers pricing and admission control in a system with customer abandonment. These features yield a fundamentally different control problem, which we solve in closed form.

We draw on and contribute to the literature on the dynamic control of queueing systems in heavy traffic. This line of work, pioneered by Harrison (1988, 2000, 2003), approximates complex queueing and stochastic networks with Brownian systems, yielding tractable approximations. Early examples include the papers by Harrison and Wein (1989, 1990), which study optimal sequencing for a crisscross network and a multiclass two-station closed queueing network, respectively. Since then, this approach has been used to study a variety of production systems (Plambeck et al. 2001, Plambeck 2004, Çelik and Maglaras 2008, Ata and Olsen 2009, Gao and Huang 2023, Sun and Zhu 2025) and service systems (Tezcan and Dai 2010, Ata and Tongarlak 2013, Ghamami and Ward 2013, Kim et al. 2018, Ata et al. 2020). More broadly, diffusion approximations have been used to study a range of related problems, including staffing (Hong et al. 2023) and optimal investment timing (Gao et al. 2025).

In some formulations, heavy traffic approximations lead to drift rate control problems in the Brownian system. A seminal paper is Ata et al. (2005), which studies drift control of a reflected Brownian motion on a bounded interval; Ghosh and Weerasinghe (2007) extend this paper by incorporating nondecreasing holding costs and an endogenously chosen upper barrier. Related formulations have since been used to study admission control (Ata 2006), scheduling in systems with abandonment (Rubino and Ata 2009, Ghamami and Ward 2013), service and activity rate control (Ghosh and Weerasinghe 2010, Liu and Sun 2022, Ata et al. 2024), pricing (Ata and Barjesteh 2022, Alwan et al. 2024), and capacity expansion (Sun and Liu 2025). The drift rate control problem we study has a distinct structure. Specifically, the omnichannel and hybrid nature of our system induces a piecewise-linear state cost function with multiple endogenous breakpoints. Moreover, abandonments couple the state cost function with the value function. These features link the drift rate and workload configuration decisions in a way that is absent in existing formulations, and necessitate a new solution approach.

3. Model

This section introduces a model for an omnichannel hybrid production system that serves a market of heterogeneous, price- and delay-sensitive customers. The system is hybrid in that the firm produces two types of goods: make-to-order (MTO) and make-to-stock (MTS). MTO goods are produced only after an order is received, while MTS goods may be produced in advance and held as finished-goods inventory. Moreover, the system is omnichannel in that the firm offers goods to its customers through two channels: walk-in and online. Each good may be offered through one or both channels.

Goods offered through the online channel are available at multiple quote times, where a quote time is defined as the time between when the order is placed and the promised pickup time. Each online good may be offered at a different set of quote times. For notational convenience, we assign a quote time of zero to all orders placed through the walk-in channel. We assume that the menu of offered goods, their production mode, and the available quote times are predetermined and fixed throughout the paper.

We define a product as a combination of a good and a quote time. We numerically label each product and let $\mathcal{S} := \{1,\dots,K\}$ denote the set of all products, where $K \in \mathbb{N} := \{1,2,\dots\}$ denotes the total number of distinct products. For ease of exposition, we partition $\mathcal{S}$ into four subsets: walk-in MTO products $\mathcal{S}_{\text{w}}^{\text{MTO}}$, online MTO products $\mathcal{S}_{\text{o}}^{\text{MTO}}$, walk-in MTS products $\mathcal{S}_{\text{w}}^{\text{MTS}}$, and online MTS products $\mathcal{S}_{\text{o}}^{\text{MTS}}$. We also let

$$\mathcal{S}^{\text{MTO}} := \mathcal{S}_{\text{w}}^{\text{MTO}} \cup \mathcal{S}_{\text{o}}^{\text{MTO}}, \quad \mathcal{S}^{\text{MTS}} := \mathcal{S}_{\text{w}}^{\text{MTS}} \cup \mathcal{S}_{\text{o}}^{\text{MTS}}, \quad \mathcal{S}_{\text{w}} := \mathcal{S}_{\text{w}}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}, \quad \mathcal{S}_{\text{o}} := \mathcal{S}_{\text{o}}^{\text{MTO}} \cup \mathcal{S}_{\text{o}}^{\text{MTS}}$$

denote the set of MTO products, MTS products, walk-in products, and online products, respectively. We assume that the firm offers at least one MTS product (i.e., $\mathcal{S}^{\text{MTS}} \neq \emptyset$) and that all MTS goods are available through the walk-in channel.[2] For $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$, we let $w(k) \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ denote the walk-in MTS product that shares the same good as online MTS product k. Conversely, we let

$$\mathcal{S}_{\text{o}}^{\text{MTS}}(k) := \{j \in \mathcal{S}_{\text{o}}^{\text{MTS}} : w(j) = k\}, \quad k \in \mathcal{S}_{\text{w}}^{\text{MTS}},$$

denote the set of online MTS products that share the same good as walk-in MTS product k.

The quote time of product $k \in \mathcal{S}$ is denoted by $\delta_k \in \mathbb{R}$, where $\delta_k > 0$ for online products $k \in \mathcal{S}_{\text{o}}$ and $\delta_k = 0$ for walk-in products $k \in \mathcal{S}_{\text{w}}$. We assume that customers ordering product k, also referred to as class k customers, arrive to pick up their goods precisely δ_k time units after placing their order.

The system manager controls the system by making dynamic pricing, scheduling, and order rejection decisions. Section 3.1 discusses the demand structure and pricing decisions. Section 3.2 introduces our stochastic processing network model and discusses the scheduling and rejection decisions. Section 3.3 discusses the state dynamics. Section 3.4 discusses the cost structure and the stochastic control problem.

3.1. Demand Structure

We assume customers arrive according to a nonhomogeneous Poisson process, whose instantaneous intensity depends on the prices set by the system manager. Formally, let $\mathcal{P} \subset \mathbb{R}_+^K$ denote the set of admissible price vectors. The system manager chooses a price vector $p(t) = (p_1(t),\dots,p_K(t)) \in \mathcal{P}$ at each time t, where $p_k(t)$ denotes the price of product k at time t. The corresponding demand rates are determined by a demand function $\Lambda : \mathcal{P} \to \mathbb{R}_+^K$ that maps the price vector to an instantaneous demand rate vector. The instantaneous demand rate vector at time t is denoted by $\lambda(t) = \Lambda(p(t))$, where $\lambda_k(t)$ denotes the instantaneous demand rate of product k at time t. The set of admissible instantaneous demand rate vectors is denoted by $\mathcal{L} := \{\Lambda(p) : p \in \mathcal{P}\} \subseteq \mathbb{R}_+^K$. We refer to $p = \{p(t) : t \geq 0\}$ as the price process and to $\lambda = \{\lambda(t) : t \geq 0\}$ as the instantaneous demand rate process. To facilitate the analysis, we impose the following regularity assumption:

ASSUMPTION 1. *There exists an inverse demand function $\Lambda^{-1} : \mathcal{L} \to \mathcal{P}$.*

This assumption is satisfied by many common demand functions, such as linear, exponential, and logit demand models. Under this assumption, we can equivalently treat the instantaneous demand process as the system manager's control, since any $\lambda(t) \in \mathcal{L}$ can be implemented by setting $p(t) = \Lambda^{-1}(\lambda(t)) \in \mathcal{P}$.

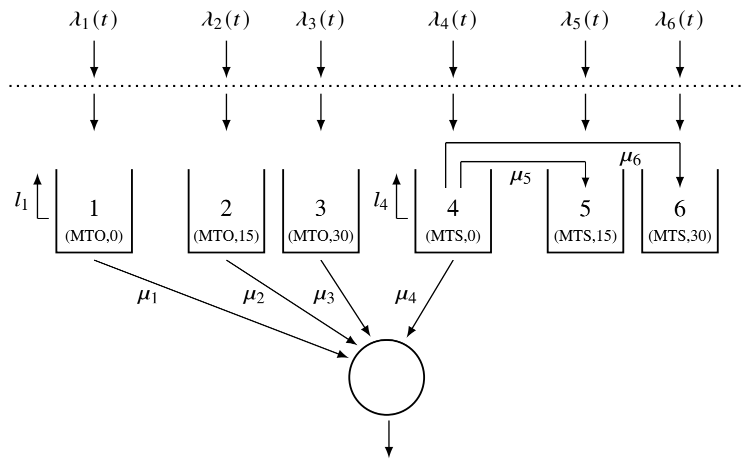


Figure 1 Our stochastic processing network for a six-product example with one MTO good, one MTS good, and two quote times (15 and 30 minutes). Each buffer is labeled with its product index, production mode, and quote time. Products 1 and 4 are walk-in products; products 2, 3, 5, and 6 are online products. Activities 1–4 represent production; activities 5–6 represent reallocation. The dotted line represents order rejections and the upward arrows represent abandonments.

Observe that the arrival rates in our model may depend on the full price vector, thereby capturing substitution effects across products, including substitution across channels and quote times. In particular, the demand model accommodates cross-price elasticities and thus captures the shift in demand across channels and quote times in response to dynamic pricing. Although we assume that the prices of all products can be dynamically adjusted, this framework also naturally accommodates limited pricing flexibility. In the numerical study, we consider two practical variants: static pricing and online-only dynamic pricing.

3.2. Stochastic Processing Network Formulation

We model the omnichannel hybrid production system as a single-server stochastic processing network (Harrison 2000, 2003); see Figure 1 for an illustration of a six-product example in our setting. A stochastic processing network generalizes a traditional queueing network and consists of three basic elements: jobs, servers, and activities. The notions of jobs and servers are identical to those in a queueing network. In our setting, jobs correspond to product orders and the server represents the production system. Each product class has a dedicated buffer where the orders of that class are stored.

The distinctive feature of a stochastic processing network is the notion of an activity, which allows the model to accommodate more complex operational tasks beyond those captured in a traditional queueing network; to be more specific, an activity need not be confined to a single server serving a single buffer. In our setting, activities correspond to the system manager's order scheduling decisions. These activities fall into two distinct categories: production and reallocation. There is a one-to-one correspondence between activities and product classes. Specifically, for $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}$, activity k is a production activity, whereas for $k \in \mathcal{S}^{\text{MTS}}_{\text{o}}$, activity k is a reallocation activity. We describe these two types of activities below.

Production Activities. Production activities represent the operational task of producing goods. These activities are associated with MTO products and walk-in MTS products. For $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}$, activity k

produces one unit of the good associated with product k. Note that online MTS products do not have dedicated production activities. Instead, to satisfy online MTS demand, the associated goods are produced through the production activity of the corresponding walk-in MTS product and then assigned to online orders via reallocation activities, as described below.

The duration of production activity $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$, i.e., the production time of the good associated with product k, follows a general distribution with mean $m_k > 0$ and coefficient of variation $c_{sk} > 0$. Let $\mu_k := 1/m_k$ denote the production rate for product k, and let $S_k(t)$ denote the total number of class k products produced if the system were to continuously engage in activity k up to time t. Since online MTS products are fulfilled using the inventory of the corresponding walk-in MTS products, we extend the notion of the mean production time to online MTS products by setting $m_k := m_{w(k)}$ for $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$. The K-dimensional mean production time vector is defined as $m = (m_1, \ldots, m_K)$.

Reallocation Activities. Reallocation activities represent the operational task of fulfilling outstanding online MTS orders using available inventory from the corresponding walk-in MTS buffer. For $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$, activity k reallocates inventory from the walk-in buffer $w(k)$ to satisfy outstanding orders for product k. We assume reallocation activities are instantaneous; that is, they require no processing time. Let $\mu_k > 0$ denote the number of products reallocated per unit of reallocation activity k. That is, one good is reallocated from its walk-in MTS buffer to online MTS buffer k for every $1/\mu_k$ units of reallocation activity k undertaken. (Allowing $\mu_k \neq 1$ facilitates the analysis but is otherwise inconsequential.) Let $S_k(t) := \lfloor \mu_k t \rfloor$ denote the number of products reallocated as a result of the first t units of reallocation activity k undertaken.[3]

Reallocation activities are essential because they capture the inherent fungibility of MTS goods, enabling them to satisfy demand in any channel or quote time. In a formulation without reallocation activities, such as one with separate production activities for each online MTS product, the system manager must commit inventory ex ante to a specific product class at the time of production. This can leave inventory stranded in one channel while unmet demand accumulates in another, leading to avoidable holding and tardiness costs.

Dynamic Control Policy. The system manager makes pricing, scheduling, and order rejection decisions. The pricing decisions are captured by the instantaneous demand process λ; see Section 3.1. Scheduling decisions determine when and how much to engage in production and reallocation activities. A scheduling policy is represented by a K-dimensional allocation process $T = \{T(t) : t \geq 0\}$, where $T_k(t)$ tracks the cumulative "effort" devoted to activity $k \in \mathcal{S}$ up to time t. That is, for $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$, $T_k(t)$ denotes the cumulative time devoted to producing product k up to time t, whereas for $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$, $T_k(t)$ denotes the cumulative units of reallocation activity k undertaken up to time t. We allow preemptive-resume service and restrict attention to head-of-line scheduling policies. Under a scheduling policy T, the cumulative server idleness up to time t is defined as follows:

$$I(t) := t - \sum_{k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}} T_k(t), \quad t \geq 0. \tag{1}$$

We refer to $I = \{I(t) : t \geq 0\}$ as the cumulative idleness process. Note that production activities require time and consume server capacity, whereas reallocation activities are instantaneous and do not use server capacity.

Rejection decisions capture the system manager's ability to reject orders at a fixed cost to mitigate excessive waiting and abandonment costs. A rejection policy is represented by a K-dimensional process $R = \{R(t) : t \geq 0\}$, where $R_k(t)$ denotes the cumulative number of rejected product $k \in \mathcal{S}$ orders up to time t.

Accordingly, the system manager's dynamic control policy is a triple (λ, T, R), where λ is the instantaneous demand rate process, R is the rejection process, and T is the allocation process.

3.3. State Dynamics

We now describe the state dynamics under a fixed control policy (λ, T, R). As discussed in Section 3.1, demand for product $k \in \mathcal{S}$ arrives according to a nonhomogeneous Poisson process with instantaneous rate $\lambda_k(t)$. Thus, the cumulative demand for product k up to time t is given by $N_k\big(\int_0^t \lambda_k(s)\, ds\big)$, where N_k is a unit-rate Poisson process. The cumulative number of accepted orders for product k up to time t is

$$A_k(t) := N_k\Big(\int_0^t \lambda_k(s)\, ds\Big) - R_k(t), \quad t \geq 0, \tag{2}$$

where the two terms respectively denote the cumulative number of product k orders received and rejected up to time t. We refer to $A = \{A(t) : t \geq 0\}$ as the accepted orders process. Walk-in customers are impatient and may abandon when faced with excessive waiting times, whereas online customers are assumed not to abandon. For $k \in \mathcal{S}_{\text{w}}$, we assume that class k customers abandon the system after an exponentially distributed amount of time with rate $\ell_k \geq 0$. We denote by M_k the unit-rate Poisson process used to generate abandonments for class $k \in \mathcal{S}_{\text{w}}$. We assume $\{N_k\}_{k \in \mathcal{S}}$, $\{M_k\}_{k \in \mathcal{S}_{\text{w}}}$, and $\{S_k\}_{k \in \mathcal{S}}$ are mutually independent.

The system state is described by the K-dimensional queue length process $Q = \{Q(t), t \geq 0\}$, where $Q_k(t)$ denotes the number of outstanding orders for product $k \in \mathcal{S}$ at time t. Assuming that the system is initially empty, i.e., $Q(0) = 0$, the queue length process evolves as follows: For MTO products $k \in \mathcal{S}^{\text{MTO}}$,

$$Q_k(t) = N_k\Big(\int_0^t \lambda_k(s)\, ds\Big) - S_k\big(T_k(t)\big) - M_k\Big(\int_0^t \mathbf{1}_{\{k \in \mathcal{S}_{\text{w}}^{\text{MTO}}\}} \ell_k Q_k^+(s)\, ds\Big) - R_k(t), \quad t \geq 0, \tag{3}$$

where the four terms respectively denote the cumulative number of class k orders received, products produced, customer abandonments, and orders rejected up to time t.[4] For walk-in MTS products $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$,

$$Q_k(t) = N_k\Big(\int_0^t \lambda_k(s)\, ds\Big) - S_k\big(T_k(t)\big) - M_k\Big(\int_0^t \ell_k Q_k^+(s)\, ds\Big) - R_k(t) + \sum_{j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k)} S_j\big(T_j(t)\big), \quad t \geq 0, \tag{4}$$

where the first four terms are analogous to those in (3). The fifth term denotes the cumulative number of class k products reallocated to an online MTS buffer (sharing the same good as product k) up to time t. Note that since MTS goods can be produced in advance, $Q_k(t)$ can potentially be negative, reflecting a positive inventory of the good. For online MTS products $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$,

$$Q_k(t) = N_k\Big(\int_0^t \lambda_k(s)\, ds\Big) - S_k\big(T_k(t)\big) - R_k(t), \quad t \geq 0, \tag{5}$$

where the first and third terms are analogous to those described in (3) and (4). The second term denotes the cumulative number of class $w(k)$ products reallocated to buffer k up to time t.

A dynamic control policy (λ, T, R) is said to be admissible if it is nonanticipating and satisfies

$$I \text{ is nondecreasing and continuous with } I(0) = 0, \tag{6}$$

$$A,\ R, \text{ and } T \text{ are nondecreasing with } A(0) = R(0) = T(0) = 0, \tag{7}$$

$$\lambda(t) \in \mathcal{L} \text{ for all } t \geq 0, \tag{8}$$

$$\int_0^\infty \mathbf{1}_{\{Q_k(t) \geq 0\}}\, dT_j(t) = 0 \text{ for all } k \in \mathcal{S}_{\text{w}}^{\text{MTS}} \text{ and } j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k), \tag{9}$$

$$Q_k(t) \geq 0 \text{ for all } k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{o}}^{\text{MTS}} \text{ and } t \geq 0. \tag{10}$$

Equations (6)–(7) and (10) describe the inherent physical limitations governing the dynamics of A, I, R, T, and Q. In particular, (6) ensures that the total time allocated to production activities over any finite interval does not exceed the available time, i.e.,

$$\sum_{k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}} (T_k(t) - T_k(s)) \leq t - s \quad \text{for all} \quad 0 \leq s \leq t < \infty.$$

Equation (8) ensures that the instantaneous demand rate is achievable. Equation (9) ensures that reallocation from a walk-in MTS buffer may only occur when inventory is available in that buffer. Finally, (10) enforces that MTO products cannot be produced in advance and, without loss of generality, that inventory is reallocated to an online MTS buffer only when it is used to immediately satisfy an outstanding online order.

3.4. Stochastic Control Problem

We next formulate the stochastic control problem. We first introduce the model cost parameters, including production, tardiness, earliness, holding, abandonment, and rejection costs, and then we define the objective function. Let $\gamma_k > 0$ denote the variable cost of production for product $k \in \mathcal{S}$, and let $\gamma = (\gamma_1, \ldots, \gamma_K)$.[5] We define the instantaneous profit rate function $\Pi : \mathcal{L} \to \mathbb{R}$ as follows:

$$\Pi(x) := x'\big(\Lambda^{-1}(x) - \gamma\big), \quad x \in \mathcal{L}. \tag{11}$$

This function represents the revenue rate net of the variable cost of production in an idealized setting absent congestion-related concerns and stochastic variability, assuming that all orders are accepted and fulfilled.

We next introduce tardiness and earliness costs. Tardiness costs are incurred when an order is not ready upon the customer's arrival, whereas earliness costs are incurred when an order is completed prior to the customer's arrival. To capture these costs parsimoniously, for $k \in \mathcal{S}$, we define the cost function

$$\upsilon_k(x) := \begin{cases} \alpha_k x, & x \geq 0, \\ -\beta_k x, & x < 0, \end{cases} \tag{12}$$

where $\alpha_k > 0$ and $\beta_k \geq 0$ represent the per-unit tardiness and earliness costs for product k, respectively. Tardiness costs apply to all products. Earliness costs, however, arise only for online MTO products, since completing their production too early can lead to quality degradation and customer dissatisfaction. They do not apply to walk-in MTO products because their production cannot begin prior to the customer's arrival, and they do not apply to MTS products because these products are intended to be pre-produced by design. Thus, we set $\beta_k = 0$ for $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}} \cup \mathcal{S}^{\mathrm{MTS}}$, while $\beta_k > 0$ for $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}$.

To compute the earliness or tardiness cost for an order, we must compare the time at which the order is completed with the customer's arrival time. Let $w_k(t)$ denote the (virtual) sojourn time for a product $k \in \mathcal{S}$ order arriving at time t. It follows that the order is early if $w_k(t) < \delta_k$ and tardy if $w_k(t) \geq \delta_k$. Thus, by (12), the combined earliness and tardiness cost incurred for a product k order arriving at time t is given by $\upsilon_k(w_k(t) - \delta_k)$. In a deterministic setting, the firm could perfectly align the sojourn time of an online order with its quote time, thereby avoiding both earliness and tardiness costs. However, in the presence of stochastic variability, such an alignment becomes a nontrivial operational challenge.

We next introduce holding, abandonment, and rejection costs. Holding costs are incurred whenever there is on-hand finished-goods inventory in the walk-in MTS buffers; let $h_k > 0$ for $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$ denote the per-unit holding cost for class k products. Abandonment costs are incurred when walk-in customers abandon the system; let $d_k > 0$ for $k \in \mathcal{S}_{\mathrm{w}}$ denote the abandonment cost for class k customers. Rejection costs are incurred when the firm rejects product orders; let $r_k > 0$ for $k \in \mathcal{S}$ denote the rejection cost for a class k order.

Given an admissible policy (λ, T, R), the cumulative profit $V(t)$ earned up to time t is defined as

$$V(t) := \int_0^t \Pi\big(\lambda(s)\big)\, ds - \sum_{k\in\mathcal{S}} \int_0^t \upsilon_k\big(w_k(s) - \delta_k\big)\, dA_k(s) - \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k Q_k^-(s)\, ds - \sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k M_k\Big(\int_0^t \ell_k Q_k^+(s)\, ds\Big) - \sum_{k\in\mathcal{S}} r_k R_k(t), \quad t \geq 0, \tag{13}$$

where the first term is a surrogate for the revenue minus variable production costs and the second through fifth terms capture, respectively, the earliness and tardiness, holding, abandonment, and rejection costs. We refer to $V = \{V(t) : t \geq 0\}$ as the cumulative profit process. We next impose the following natural assumption, which ensures that it is less costly to reject a customer up front than to risk customer dissatisfaction (due to a long waiting time) and an eventual abandonment.

ASSUMPTION 2. *For $k \in \mathcal{S}_{\mathrm{w}}$, the rejection cost r_k is less than or equal to the abandonment cost d_k.*

Adopting the long-run average expected profit criterion, the system manager seeks to find an admissible dynamic control policy (λ, T, R) so as to

$$\text{maximize} \liminf_{t\to\infty} \frac{1}{t}\, \mathbb{E}\big[V(t)\big] \text{ subject to (1)–(10).} \tag{14}$$

Unfortunately, this formulation is a nonlinear, multidimensional stochastic control problem. Furthermore, when production times are not exponentially distributed, it is non-Markovian. As a result, the stochastic control problem is analytically intractable and suffers from the curse of dimensionality unless the total number of products is small. Therefore, we next consider a sequence of systems in the (conventional) heavy-traffic regime and formulate an approximating Brownian control problem that is analytically tractable.

4. Approximating Brownian Control Problem

This section introduces a Brownian approximation for the control problem discussed in Section 3, which emerges as the limit of a sequence of scaled systems under conditions of heavy traffic. The key assumption underlying the approximation is that the production and instantaneous demand rates are sufficiently large. This procedure allows us to interpret the solution of the Brownian control problem in the context of the original control problem and, in turn, use it to guide the design of our proposed policy.

4.1. Heavy Traffic Assumption and Asymptotic Regime

We consider the following static planning problem (SPP), which ignores randomness in the system:

Static Planning Problem. Choose an instantaneous demand rate vector λ so as to

$$\text{maximize } \Pi(\lambda) \text{ subject to } \lambda \in \mathcal{L}. \tag{15}$$

This problem maximizes the profit rate subject to the instantaneous demand rate vector being achievable. If the system manager were to disregard randomness and all congestion-related costs, they would use an instantaneous demand rate that solves the SPP (15). However, due to randomness, they may benefit from dynamic adjustments to the instantaneous demand rate, which are captured by the Brownian approximation.

To develop the Brownian approximation, we consider a sequence of systems indexed by a system parameter $n \in \mathbb{N}$. We attach a superscript n to various quantities in the nth system in this sequence. We focus on the asymptotic regime where both demand and system capacity grow with n as follows: for $x \in \mathcal{P}$ and $n \in \mathbb{N}$,

$$\Lambda^n(x) := n\Lambda(x) \quad \text{and} \quad \mu^n := n\mu + \sqrt{n}\eta, \tag{16}$$

where $\mu, \eta \in \mathbb{R}_+^K$ are given constants. It follows from (16) that for $n \in \mathbb{N}$ and $x \in \mathcal{L}$,

$$(\Lambda^n)^{-1}(nx) = \Lambda^{-1}(x) \quad \text{and} \quad \Pi^n(nx) = n\Pi(x). \tag{17}$$

ASSUMPTION 3 **(Heavy Traffic Assumption).** *The SPP (15) has a unique optimal solution $\lambda^\star$. Moreover, $\lambda^\star \in interior(\mathcal{L})$ and $\sum_{k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}} \rho_k = 1$, where ρ is the server utilization vector defined as follows:*

$$\rho_k := \begin{cases} \lambda_k^\star / \mu_k, & k \in \mathcal{S}^{\text{MTO}}, \\ \left(\lambda_k^\star + \sum_{j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k)} \lambda_j^\star\right) / \mu_k, & k \in \mathcal{S}_{\text{w}}^{\text{MTS}}. \end{cases} \tag{18}$$

We refer to $\lambda^\star$ as the nominal instantaneous demand rate. Assumption 3 states that the nominal demand rate $\lambda^\star$ fully utilizes system capacity, thereby placing the system in heavy traffic. For $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}$, we interpret ρ_k as the proportion of time that, in the absence of randomness, should be allocated to producing class k products to meet the nominal demand. Note that for $k \in \mathcal{S}^{\text{MTS}}_{\text{w}}$, the term ρ_k accounts for the combined demand from both the walk-in and online channels, as online MTS products are not produced directly. To ensure analytical tractability, we next impose the following regularity condition.

ASSUMPTION 4. Π *is twice continuously differentiable and strictly concave in a neighborhood of* $\lambda^\star$.

It follows from Assumption 3 and (17) that, under a deterministic (fluid) approximation, the system manager would choose the instantaneous demand rate $n\lambda^\star$. In the stochastic system, however, it may be beneficial to dynamically adjust the instantaneous demand rate around $n\lambda^\star$ in response to congestion. Because queue lengths in heavy traffic are of order $\sqrt{n}$, fluctuations in the demand rate should also be of order $\sqrt{n}$ to effectively manage congestion. (Adjustments of smaller order are asymptotically negligible, while adjustments of a larger order would override the queue length dynamics in the limit.) Therefore, we focus on instantaneous demand rate vectors of the following form: for $n \in \mathbb{N}$ and some $\zeta : [0,\infty) \to \mathbb{R}^K$,

$$\lambda^n(t) = n\lambda^\star + \sqrt{n}\,\zeta(t), \quad t \geq 0; \tag{19}$$

see Çelik and Maglaras (2008) and Ata and Barjesteh (2022) for similar treatments. The process ζ captures the demand (or pricing) control, which includes all variants of our proposed policy.

As discussed above, since queue lengths in the heavy traffic regime are of order $\sqrt{n}$, rejections should also be of the same order to effectively regulate congestion. Accordingly, we define the scaled queue length process $Z^n = \{Z^n(t) : t \geq 0\}$ and the scaled rejection process $O^n = \{O^n(t) : t \geq 0\}$ as follows:

$$Z^n(t) := \frac{Q^n(t)}{\sqrt{n}} \quad \text{and} \quad O^n(t) := \frac{R^n(t)}{\sqrt{n}}, \quad t \geq 0. \tag{20}$$

Furthermore, as thoroughly argued in Harrison (1988, 2000, 2003), any dynamic allocation policy worthy of consideration should satisfy the following for all $t \geq 0$ when n is large:

$$T^n_k(t) \approx \begin{cases} \rho_k t, & k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}, \\ \lambda^\star_k t/\mu_k, & k \in \mathcal{S}^{\text{MTS}}_{\text{o}}. \end{cases} \tag{21}$$

That is, for $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}$, ρ_k should give a first-order approximation to the fraction of time allocated to producing class k products, and for $k \in \mathcal{S}^{\text{MTS}}_{\text{o}}$, $\lambda^\star_k/\mu_k$ should give a first-order approximation to the amount of reallocation activity undertaken per unit of time. However, the system manager may benefit from making second-order deviations from these nominal rates. Thus, we define the centered and scaled allocation process $Y^n = \{Y^n(t) : t \geq 0\}$ and the scaled cumulative idleness process $L^n = \{L^n(t) : t \geq 0\}$ as follows:

$$Y^n_k(t) := \begin{cases} \sqrt{n}\big(\rho_k t - T^n_k(t)\big), & k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}, \\ \sqrt{n}\big(\lambda^\star_k t/\mu_k - T^n_k(t)\big), & k \in \mathcal{S}^{\text{MTS}}_{\text{o}}, \end{cases} \quad \text{and} \quad L^n(t) := \sqrt{n} I^n(t) = \sum_{k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}} Y^n_k(t), \quad t \geq 0. \tag{22}$$

4.2. Brownian Control Problem

As discussed above, in heavy traffic, congestion-related quantities are of order $\sqrt{n}$, so the per-unit costs are scaled to keep the various cost components comparable and prevent any component from vanishing or dominating in the limit. We validate the robustness of this scaling via a sensitivity analysis in Section 8. To that end, we assume that the tardiness costs α_k^n, earliness costs β_k^n, holding costs h_k^n, abandonment costs d_k^n, and rejection costs r_k^n (for the product classes to which they apply) scale with n as follows:

$$\alpha_k^n := \frac{\alpha_k}{\sqrt{n}}, \quad \beta_k^n := \frac{\beta_k}{\sqrt{n}}, \quad h_k^n := \frac{h_k}{\sqrt{n}}, \quad d_k^n := \frac{d_k}{\sqrt{n}}, \quad \text{and} \quad r_k^n := \frac{r_k}{\sqrt{n}}, \tag{23}$$

where $\alpha_k, \beta_k, h_k, d_k$, and r_k are nonnegative constants; see Ata and Barjesteh (2022) and Gao et al. (2023a) for similar treatments. Motivated by (12) and (23), for $k \in \mathcal{S}$, we define the scaled cost function

$$\upsilon_k^n(x) := \frac{\upsilon_k(x)}{\sqrt{n}} = \begin{cases} \alpha_k^n x, & x \geq 0, \\ -\beta_k^n x, & x < 0. \end{cases} \tag{24}$$

Since queue lengths are of order $\sqrt{n}$ in heavy traffic, while the system capacity is of order n, the sojourn times are of order $1/\sqrt{n}$. Therefore, we assume the quote times δ_k^n scale with n as follows:

$$\delta_k^n := \frac{\delta_k}{\sqrt{n}}, \quad k \in \mathcal{S}, \tag{25}$$

where δ_k is positive for online products and zero for walk-in products; see Rubino and Ata (2009) for similar a treatment. Finally, we assume the abandonment rates ℓ_k^n are as follows:

$$\ell_k^n := \ell_k, \quad k \in \mathcal{S}_{\text{w}}, \tag{26}$$

where ℓ_k are nonnegative constants. If the system manager were to ignore stochastic variability, and thus choose the instantaneous demand rate $n\lambda^\star$ for the nth system, the cumulative profit up to time t would be $n\Pi(\lambda^\star)t$, which serves as an upper bound for the cumulative profit in the stochastic system.

PROPOSITION 1. *Under any admissible dynamic control policy (λ^n, T^n, R^n), we have that $V^n(t) \leq n\Pi(\lambda^\star)t$ for all $t \geq 0$ and sufficiently large $n \in \mathbb{N}$.*

Motivated by Proposition 1, we define the cumulative cost process $\xi^n = \{\xi^n(t) : t \geq 0\}$ as the deviation of the cumulative profit process V^n from that of the corresponding deterministic system:

$$\xi^n(t) := n\Pi(\lambda^\star)t - V^n(t), \quad t \geq 0. \tag{27}$$

Appendix EC.1 formally derives the Brownian approximation as n gets large. In this approximation, the processes Z^n, Y^n, L^n, O^n, and ξ^n are replaced by their formal limits Z, Y, L, O, and ξ, respectively. The system manager's dynamic control policy in the Brownian approximation takes the form of a triple (ζ, Y, O), where ζ represents the dynamic adjustments to the pricing policy, Y represents the dynamic adjustments to the scheduling policy, and O represents the dynamic order rejection policy. The K-dimensional process

$Z = \{Z(t) : t \geq 0\}$ describes the limiting queue length dynamics in the Brownian system, induced by abandonments and pricing, scheduling, and rejection controls, and evolves as follows: for $t \geq 0$,

$$Z_k(t) = X_k(t) + \int_0^t \zeta_k(s)\,ds - \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k Z_k^+(s)\,ds - O_k(t) + \mu_k Y_k(t), \quad k \in \mathcal{S}^{\mathrm{MTO}}, \tag{28}$$

$$Z_k(t) = X_k(t) + \int_0^t \zeta_k(s)\,ds - \int_0^t \ell_k Z_k^+(s)\,ds - O_k(t) + \mu_k Y_k(t) - \sum_{j\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \mu_j Y_j(t), \quad k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \tag{29}$$

$$Z_k(t) = X_k(t) + \int_0^t \zeta_k(s)\,ds - O_k(t) + \mu_k Y_k(t), \quad k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}, \tag{30}$$

where $X_k = \{X_k(t) : t \geq 0\}$ for $k \in \mathcal{S}$ are independent Brownian motions with infinitesimal drift ν_k and infinitesimal variance σ_k^2 given as follows:

$$\nu_k := \begin{cases} -\eta_k\rho_k, & k \in \mathcal{S}^{\mathrm{MTO}}, \\ -\eta_k\rho_k + \sum_{j\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \eta_j\lambda_j^\star/\mu_j, & k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \\ -\eta_k\lambda_k^\star/\mu_k, & k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}, \end{cases} \quad \text{and} \quad \sigma_k^2 := \begin{cases} \lambda_k^\star\left(1 + c_{sk}^2\right), & k \in \mathcal{S}^{\mathrm{MTO}}, \\ \lambda_k^\star + (\lambda_k^\star + \sum_{j\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \lambda_j^\star)\, c_{sk}^2, & k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \\ \lambda_k^\star, & k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}. \end{cases} \tag{31}$$

Equations (28)–(30) are the Brownian counterparts of the queue length dynamics (3)–(5). The process $L = \{L(t) : t \geq 0\}$ represents cumulative server idleness in the Brownian system and satisfies

$$L(t) := \sum_{k\in\mathcal{S}^{\mathrm{MTO}}\cup\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} Y_k(t), \quad t \geq 0. \tag{32}$$

Equation (32) is the Brownian counterpart of (1). The processes L, O, and Z must satisfy the following:

$$L \text{ and } O \text{ are nondecreasing with } L(0) = O(0) = 0, \tag{33}$$

$$Z_k(t) \geq 0 \text{ for all } k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}. \tag{34}$$

These conditions are the Brownian counterparts of the feasibility conditions (6) and (10), respectively.

The process $\xi = \{\xi(t) : t \geq 0\}$ represents the cumulative cost in the Brownian system. It consists of five terms that capture the loss in revenue associated with deviating from the nominal static prices and the costs associated with earliness and tardiness, holding, abandonment, and rejections. Specifically,

$$\begin{aligned}\xi(t) := \int_0^t \zeta(s)' H \zeta(s)\,ds + \sum_{k\in\mathcal{S}} \int_0^t \upsilon_k\big(Z_k(s) - \lambda_k^\star\delta_k\big)\,ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k Z_k^-(s)\,ds \\ + \sum_{k\in\mathcal{S}_{\mathrm{w}}} \int_0^t d_k\ell_k Z_k^+(s)\,ds + \sum_{k\in\mathcal{S}} r_k O_k(t), \quad t \geq 0,\end{aligned} \tag{35}$$

where $H := -\nabla^2\Pi(\lambda^\star)/2 \in \mathbb{R}^{K\times K}$ is a symmetric, positive definite matrix (see Assumption 4), which captures the local curvature of the instantaneous profit rate function around $\lambda^\star$, quantifying the loss in revenue associated with dynamic demand adjustments. Equation (35) is the Brownian counterpart of (13).

A dynamic control policy (ζ, Y, O) is said to be admissible if it is nonanticipating and satisfies (33)–(34) together with the following regularity condition:

$$\limsup_{t\to\infty} \frac{1}{t}\, \mathbb{E}\big[\|Z(t)\|\big] = 0, \tag{36}$$

where $\|\cdot\|$ denotes the Euclidean norm. The Brownian control problem (BCP) is as follows: Choose an admissible dynamic control policy (ζ, Y, O) so as to

$$\text{minimize} \limsup_{t\to\infty} \frac{1}{t}\, \mathbb{E}\big[\xi(t)\big] \text{ subject to (28)–(35).} \tag{37}$$

Although the Brownian control problem is simpler than the original control problem due to the continuous nature of the state dynamics, it remains a multidimensional stochastic control problem. We next develop an equivalent one-dimensional formulation that admits a closed-form solution.

5. Equivalent Workload Formulation

This section develops an equivalent one-dimensional control problem, which we refer to as the equivalent workload formulation. To that end, we define the workload process $W = \{W(t) : t \ge 0\}$ as follows:

$$W(t) \coloneqq \sum_{k\in\mathcal{S}} m_k Z_k(t), \quad t \ge 0. \tag{38}$$

The workload $W(t)$ represents the total (scaled) backlog in the system at time t, measured in units of work for the server. It aggregates the outstanding work associated with customer orders and the work stored as finished goods inventory for MTS products. Note that substituting (28)–(30) into (38) yields

$$W(t) = B(t) + \int_0^t \theta(s)\,ds - \int_0^t \sum_{k\in\mathcal{S}_\mathrm{w}} m_k \ell_k Z_k^+(s)\,ds + L(t) - U(t), \quad t \ge 0, \tag{39}$$

where

$$B(t) \coloneqq \sum_{k\in\mathcal{S}} m_k X_k(t), \quad \theta(t) \coloneqq \sum_{k\in\mathcal{S}} m_k \zeta_k(t), \quad L(t) \coloneqq \sum_{k\in\mathcal{S}^{\mathrm{MTO}}\cup\mathcal{S}_\mathrm{w}^{\mathrm{MTS}}} Y_k(t), \quad U(t) \coloneqq \sum_{k\in\mathcal{S}} m_k O_k(t), \quad t \ge 0. \tag{40}$$

The first term in (39) captures the stochastic variability inherited from the Brownian motion X; the process $B = \{B(t) : t \ge 0\}$ is a Brownian motion with infinitesimal drift $\mu \coloneqq \sum_{k\in\mathcal{S}} m_k \nu_k \le 0$ and variance $\sigma^2 \coloneqq \sum_{k\in\mathcal{S}} m_k^2 \sigma_k^2 > 0$. The second term captures the drift rate induced by the dynamic adjustments to the instantaneous demand rate process; we refer to $\theta = \{\theta(t) : t \ge 0\}$ as the effective drift rate process. The third term captures the drift rate generated by abandonments. The fourth and fifth terms represent, respectively, the cumulative (scaled) idleness and rejected work up to time t; the processes $L = \{L(t) : t \ge 0\}$ and $U = \{U(t) : t \ge 0\}$ are referred to as the effective idleness and rejection processes, respectively.

In the workload formulation, costs arise from three sources: (i) adjustments to the effective drift rate, (ii) congestion, and (iii) control actions that push the workload process. Given an effective drift rate x, the system manager seeks a feasible ζ that achieves x at minimum cost. Accordingly, we define the effective drift rate cost function $c:\mathbb{R}\to\mathbb{R}$ and the optimal drift rate function $\zeta^\star:\mathbb{R}\to\mathbb{R}^K$, respectively, as follows:

$$c(x) := \min\left\{\zeta' H\zeta : m'\zeta = x,\ \zeta\in\mathbb{R}^K\right\} \quad \text{and} \quad \zeta^\star(x) := \arg\min\left\{\zeta' H\zeta : m'\zeta = x,\ \zeta\in\mathbb{R}^K\right\}, \quad x\in\mathbb{R}. \tag{41}$$

These functions specify the optimal cost and corresponding ζ for a given effective drift rate x.

LEMMA 1. *The functions $c:\mathbb{R}\to\mathbb{R}$ and $\zeta^\star:\mathbb{R}\to\mathbb{R}^K$ can be equivalently expressed as follows:*

$$c(x) = \frac{x^2}{m'H^{-1}m} \quad \text{and} \quad \zeta^\star(x) = \frac{H^{-1}m}{m'H^{-1}m}x, \quad x\in\mathbb{R}.$$

To account for congestion costs, we define the set of admissible workload distribution vectors as follows:

$$\mathcal{A}(w) := \left\{z\in\mathbb{R}^K : m'z = w \text{ and } z_k\ge 0 \text{ for all } k\in\mathcal{S}^{\text{MTO}}\cup\mathcal{S}_{\text{o}}^{\text{MTS}}\right\}, \quad w\in\mathbb{R}. \tag{42}$$

A workload distribution $z\in\mathcal{A}(w)$ specifies how the workload w is allocated across products. Given a workload distribution $z\in\mathcal{A}(w)$, the resulting congestion costs are as follows: the combined earliness and tardiness cost $\sum_{k\in\mathcal{S}}\upsilon_k(z_k-\lambda_k^\star\delta_k)$, the holding cost $\sum_{k\in\mathcal{S}_{\text{w}}^{\text{MTS}}}h_k z_k^-$, and the abandonment cost $\sum_{k\in\mathcal{S}_{\text{w}}}d_k\ell_k z_k^+$.

Finally, we characterize the cost associated with pushing the workload process. Upward adjustments correspond to idleness and are costless. Downward adjustments are achieved through order rejections and incur cost. When pushing the workload process down, the system manager chooses the least costly product in terms of rejection cost per unit of work. Accordingly, we define the product class with the lowest rejection cost per unit of work and the effective rejection cost, respectively, as follows:

$$k^\star := \arg\min\{r_k/m_k : k\in\mathcal{S}\} \quad \text{and} \quad \kappa := r_{k^\star}/m_{k^\star}. \tag{43}$$

In the equivalent workload formulation, a control policy takes the form of a tuple (L,U,z,θ), where L, U, z, and θ capture the impact of server idleness, order rejections, scheduling, and dynamic price adjustments, respectively. We refer to $z:[0,\infty)\times\mathbb{R}\to\mathbb{R}^K$ as the workload configuration function. A control policy (L,U,z,θ) is said to be admissible to the equivalent workload formulation if it is nonanticipating and satisfies

$$W(t) = B(t) + \int_0^t \theta(s)\,ds - \int_0^t \sum_{k\in\mathcal{S}_{\text{w}}} m_k\ell_k z_k^+(s,W(s))\,ds + L(t) - U(t), \quad t\ge 0, \tag{44}$$

$$z(t,W(t))\in\mathcal{A}\big(W(t)\big), \quad t\ge 0, \tag{45}$$

$$L \text{ and } U \text{ are nondecreasing with } L(0)=U(0)=0, \tag{46}$$

$$\limsup_{t\to\infty}\frac{1}{t}\,\mathbb{E}\big[\|z(t,W(t))\|\big] = 0, \tag{47}$$

where B is a Brownian motion with infinitesimal drift μ and infinitesimal variance σ^2. We define

$$\Xi(t) := \int_0^t c\big(\theta(s)\big)\,ds + \sum_{k\in\mathcal{S}}\int_0^t \upsilon_k\big(z_k(s,W(s))-\lambda_k^\star\delta_k\big)\,ds + \sum_{k\in\mathcal{S}_{\text{w}}^{\text{MTS}}}\int_0^t h_k z_k^-(s,W(s))\,ds$$

$$+\sum_{k\in\mathcal{S}_{\mathrm{w}}}\int_0^t d_k\ell_k z_k^+(s,W(s))\,ds+\kappa U(t),\quad t\ge 0, \tag{48}$$

which captures the costs associated with adjustments to the effective drift rate, congestion, and pushing the workload process up to time t. We refer to $\Xi=\{\Xi(t):t\ge 0\}$ as the cumulative cost process. The equivalent workload formulation (EWF) is as follows: Choose an admissible control policy (L,U,z,θ) so as to

$$\text{minimize}\ \limsup_{t\to\infty}\frac{1}{t}\,\mathbb{E}\big[\Xi(t)\big]\ \text{ subject to (44)–(48).} \tag{49}$$

Proposition EC.1 in Appendix EC.3 proves the equivalence of the BCP (37) and the EWF (49).

6. Solution to the Equivalent Workload Formulation

This section solves the workload formulation. To minimize technical complexity, we restrict attention to stationary Markov control policies, under which the workload configuration $z(t,W(t))$ and the effective drift rate $\theta(t)$ at time t depend on the history only through the current system workload $W(t)$. To reflect this, in what follows, we write $z(W(t))$ and $\theta(W(t))$ in place of $z(t,W(t))$ and $\theta(t)$, respectively. The remainder of this section is organized as follows. Section 6.1 introduces a class of barrier policies. Section 6.2 introduces the associated Bellman equation. Finally, Section 6.3 solves the Bellman equation, derives an optimal solution (which is a barrier policy), and characterizes its structure.

6.1. Barrier Policies

Definition 1. Given $l,u\in\mathbb{R}$ such that $l<u$, we call an admissible control policy (L,U,z,θ) a barrier policy (with a lower barrier at l and an upper barrier at u) if it satisfies $W(t)\in[l,u]$ for all $t\ge 0$, and

$$\int_0^t \mathbf{1}_{\{W(s)>l\}}\,dL(s)=0\quad\text{and}\quad\int_0^t \mathbf{1}_{\{W(s)<u\}}\,dU(s)=0\quad\text{for all}\quad t\ge 0. \tag{50}$$

Note that the processes L and U reflect the workload process at the lower barrier l and upper barrier u, respectively. We next establish a method for computing the long-run average expected cost associated with a barrier policy. As a preliminary, let $C^n[l,u]$ for $n\in\mathbb{N}$ denote the space of real-valued functions $f:[l,u]\to\mathbb{R}$ that are n-times continuously differentiable. Then, given functions $z:\mathbb{R}\to\mathbb{R}^K$ and $\theta:\mathbb{R}\to\mathbb{R}$ and $\gamma\in\mathbb{R}$, define the second-order differential operator $\Gamma_{z,\theta}$ as follows:

$$\Gamma_{z,\theta}f(w)\coloneqq\frac{1}{2}\sigma^2 f''(w)+\Big(\mu+\theta(w)-\sum_{k\in\mathcal{S}_{\mathrm{w}}}m_k\ell_k z_k^+(w)\Big)f'(w),\quad (f,w)\in C^2[l,u]\times[l,u]. \tag{51}$$

The second-derivative term captures the stochastic variability inherited from the Brownian motion B; the first-derivative term reflects the drift rate, consisting of the drift rate of B, the effective drift rate control θ, and the drift rate induced by abandonments. Consider the following second-order differential equation:

$$\Gamma_{z,\theta}f(w)+c(\theta(w))+\sum_{k\in\mathcal{S}}\upsilon_k\big(z_k(w)-\lambda_k^\star\delta_k\big)+\sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}}h_k z_k^-(w)+\sum_{k\in\mathcal{S}_{\mathrm{w}}}d_k\ell_k z_k^+(w)=\gamma,\quad w\in[l,u], \tag{52}$$

subject to the boundary conditions

$$f'(l) = 0 \quad \text{and} \quad f'(u) = \kappa. \tag{53}$$

The next result reduces the evaluation of the long-run average cost of a barrier policy to solving an ordinary differential equation.

PROPOSITION 2. *Consider a barrier policy (L, U, z, θ) with a lower barrier at l and an upper barrier at u. If there exists $\gamma \in \mathbb{R}$ and $f \in C^2[l, u]$ that jointly satisfy (52)–(53), then*

$$\lim_{t\to\infty} \frac{1}{t}\, \mathbb{E}\Big[\int_0^t c\big(\theta(W(s))\big)\, ds + \sum_{k\in\mathcal{S}} \int_0^t \upsilon_k\big(z_k(W(s)) - \lambda_k^\star \delta_k\big)\, ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k z_k^-(W(s))\, ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}} \int_0^t d_k \ell_k z_k^+(W(s))\, ds + \kappa U(t) \Big] = \gamma.$$

6.2. Bellman Equation

Motivated by Proposition 2, this section introduces the Bellman equation associated with the workload formulation. The Bellman equation is introduced to motivate our solution approach, and its properties that we require will be proved from first principles. As a preliminary, we define the following cost function, which aggregates the various congestion costs, i.e., the earliness, tardiness, holding, and abandonment costs:

$$\varphi(z, y) := \sum_{k\in\mathcal{S}_{\mathrm{w}}} \ell_k\big(d_k - m_k y\big) z_k^+ + \sum_{k\in\mathcal{S}} \upsilon_k\big(z_k - \lambda_k^\star \delta_k\big) + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} h_k z_k^-, \quad (z, y) \in \mathbb{R}^K \times \mathbb{R}. \tag{54}$$

The Bellman equation is then given as follows: Find $l, u, \gamma \in \mathbb{R}$ and $f \in C^2[l, u]$ satisfying $l < u$ and

$$\min_{z\in\mathcal{A}(w),\, x\in\mathbb{R}} \left\{ \frac{1}{2}\sigma^2 f''(w) + (\mu + x) f'(w) + c(x) + \varphi\big(z, f'(w)\big) \right\} = \gamma, \quad w \in [l, u], \tag{55}$$

subject to the boundary conditions

$$f'(l) = f''(l) = f''(u) = 0 \quad \text{and} \quad f'(u) = \kappa. \tag{56}$$

The function f is referred to as the relative value function. Its derivative can be interpreted as the marginal cost of an additional unit of workload. The Bellman equation characterizes a candidate effective drift rate and workload distribution by balancing the marginal cost of workload against the costs of drift rate control and congestion. We interpret γ as a guess for the long-run average cost and l and u as the lower and upper reflecting barriers, respectively, to be imposed on the workload process. The first-order boundary conditions in (56) reflect the cost implications of enforcing the workload barriers, and the second-order boundary conditions, known as smooth pasting conditions, ensure that the barriers are chosen optimally.

Since the minimand in (55) is additively separable in x and z, and the feasible sets for x and z are independent, the joint minimization problem over x and z decouples. By Assumption 4 and Lemma 1,

$$\arg\min_{x\in\mathbb{R}} \big\{ x f'(w) + c(x) \big\} = -\frac{m' H^{-1} m}{2} f'(w), \quad w \in [l, u]. \tag{57}$$

To address the minimization over z, we define the effective state cost function $\phi : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\phi(w, y) := \min_{z \in \mathcal{A}(w)} \varphi(z, y), \quad w, y \in \mathbb{R}. \tag{58}$$

Note that the Bellman equation (55)–(56) is equivalent to a first-order differential equation by letting $v = f'$. Thus, it can be rewritten as follows: Find $l, u, \gamma \in \mathbb{R}$ and $v \in C^1[l, u]$ that satisfy $l < u$ and

$$\frac{1}{2}\sigma^2 v'(w) + \mu\, v(w) - \frac{m' H^{-1} m}{4} v^2(w) + \phi(w, v(w)) = \gamma, \quad w \in [l, u], \tag{59}$$

subject to the boundary conditions

$$v(l) = v'(l) = v'(u) = 0 \quad \text{and} \quad v(u) = \kappa. \tag{60}$$

6.3. Solution to the Bellman Equation

In this section, we establish that the Bellman equation admits a unique solution and leverage it to construct a candidate barrier policy. We then prove that this candidate policy is optimal for the equivalent workload formulation, and use it to derive structural insights into how to regulate demand and distribute workload.

To characterize the solution to the Bellman equation, we begin by introducing two workload thresholds that play a central role in the structure of the optimal policy. These thresholds arise from the quote times and govern the shape of the optimal workload configuration. To that end, let

$$w_0 := \sum_{k \in \mathcal{S}_{\text{o}}^{\text{MTO}}} m_k \lambda_k^\star \delta_k \quad \text{and} \quad w_1 := \sum_{k \in \mathcal{S}_{\text{o}}} m_k \lambda_k^\star \delta_k, \tag{61}$$

with the convention that $w_0 := 0$ when $\mathcal{S}_{\text{o}}^{\text{MTO}} = \varnothing$ and $w_1 := 0$ when $\mathcal{S}_{\text{o}} = \varnothing$. By construction, $0 \le w_0 \le w_1$. We interpret w_0 (resp., w_1) as the total workload in all online MTO classes (resp., all online classes) such that, under the nominal capacity allocation ρ, the head-of-line order in each class has a sojourn time equal to its quote time. These thresholds partition the workload space into three regions. For $w < w_0$, earliness or holding costs are unavoidable. For $w_0 \le w \le w_1$, congestion can be absorbed without incurring state costs. For $w > w_1$, tardiness or abandonment costs are unavoidable. Thus, w_0 and w_1 mark the transitions between low-, intermediate-, and high-congestion regions, which have distinct economic implications. The following theorem establishes existence, uniqueness, and key properties of the solution to the Bellman equation.

THEOREM 1. *The Bellman equation (59)–(60) has a unique solution $(l^\star, u^\star, \gamma^\star, v^\star)$. Moreover, $\gamma^\star > 0$, $l^\star < w_0 \le w_1 < u^\star$, and $v^\star \in C^1[l^\star, u^\star]$ is nonnegative and strictly increasing.*

In Appendices EC.2 and EC.4, we prove Theorem 1 and provide a closed-form solution. The monotonicity of $v^\star$ implies that the marginal cost of workload is increasing, reflecting the growing difficulty of managing additional congestion. To recover a solution to the original Bellman equation (55)–(56), we let

$$f^\star(w) := \int_{l^\star}^{w} v^\star(x)\, dx, \quad w \in [l^\star, u^\star]. \tag{62}$$

COROLLARY 1. *The tuple $(l^\star, u^\star, \gamma^\star, f^\star)$ is the unique solution (up to an additive constant in $f^\star$) to the Bellman equation (55)–(56). Moreover, $f^\star \in C^2[l^\star, u^\star]$ is strictly increasing and strictly convex.*

Equipped with a solution to the Bellman equation, we propose a candidate barrier policy for the EWF (49). In accordance with (57), we define the candidate effective drift rate as follows:

$$\theta^\star(w) \coloneqq -\frac{m' H^{-1} m}{2} v^\star(w), \quad w \in [l^\star, u^\star]. \tag{63}$$

The candidate workload configuration is a minimizer of the right-hand side of (58) when $y = v^\star(w)$, and is not unique in general. To be more specific, our candidate workload configuration function is given by

$$z^\star(w) \coloneqq \mathcal{Z}(w, v^\star(w)) \in \operatorname*{arg\,min}_{z \in \mathcal{A}(w)} \varphi(z, v^\star(w)), \quad w \in [l^\star, u^\star], \tag{64}$$

where $\mathcal{Z}$ is given in closed form by (EC.40)–(EC.42) in Appendix EC.2.2.1. To build intuition into the structure of $z^\star$, we rewrite the effective state cost function φ, defined in (54), so that all congestion costs for each product are consolidated into a single piecewise-linear term. Specifically, analogous to (12), we define

$$\hat{\alpha}_k(y) \coloneqq \begin{cases} \alpha_k + \ell_k(d_k - m_k y), & k \in \mathcal{S}_\mathrm{w}, \\ \alpha_k, & k \in \mathcal{S}_\mathrm{o}, \end{cases} \quad \text{and} \quad \hat{\beta}_k \coloneqq \begin{cases} \beta_k, & k \in \mathcal{S}_\mathrm{o} \cup \mathcal{S}_\mathrm{w}^{\mathrm{MTO}}, \\ h_k, & k \in \mathcal{S}_\mathrm{w}^{\mathrm{MTS}}, \end{cases} \tag{65}$$

for $y \in \mathbb{R}$, where $\hat{\alpha}_k(y)$ combines the tardiness and abandonment costs, and $\hat{\beta}_k$ combines the earliness and holding costs. We then define the corresponding cost functions $\hat{v}_k : \mathbb{R}^2 \to \mathbb{R}$ for $k \in \mathcal{S}$ as follows:

$$\hat{v}_k(x, y) \coloneqq \begin{cases} \hat{\alpha}_k(y)\, x, & x \geq 0, \\ -\hat{\beta}_k\, x, & x < 0. \end{cases} \tag{66}$$

It then follows from (54) and (65)–(66) that φ can be rewritten as follows:

$$\varphi(z, y) = \sum_{k \in \mathcal{S}} \hat{v}_k(z_k - \lambda_k^\star \delta_k, y) = \sum_{k \in \mathcal{S}_\mathrm{o}} \hat{v}_k(z_k - \lambda_k^\star \delta_k, y) + \sum_{k \in \mathcal{S}_\mathrm{w}} \hat{v}_k(z_k, y), \quad (z, y) \in \mathbb{R}^K \times \mathbb{R}. \tag{67}$$

In this reformulation, the workload distribution problem reduces to allocating workload across classes in a cost-efficient manner, measured by the per-unit cost functions $\hat{v}_k$.

Figures 2–3 illustrate the structure of the candidate workload configuration and resulting effective state cost. Notably, the structure of the workload configuration differs across the intervals $[l^\star, w_0)$, $[w_0, w_1]$, and $(w_1, u^\star]$, which correspond to the low-, intermediate-, and high-congestion regions. Across all regions, workload distribution follows a threshold-based structure: it is first allocated to the online classes to match their quote times, and any remaining imbalance is resolved in the least costly manner per unit of work.

Intermediate-Congestion Region. For $w \in [w_0, w_1]$, the workload is held entirely in the online classes with an effective state cost of zero; see Figure 3. In this region, a workload of $m_k \lambda_k^\star \delta_k$ is held in each online MTO class $k \in \mathcal{S}_\mathrm{o}^{\mathrm{MTO}}$, thereby holding w_0 across all online MTO classes. The remaining workload, $w - w_0$, is held in online MTS classes, with each class $k \in \mathcal{S}_\mathrm{o}^{\mathrm{MTS}}$ holding no more than $m_k \lambda_k^\star \delta_k$. No workload is held in

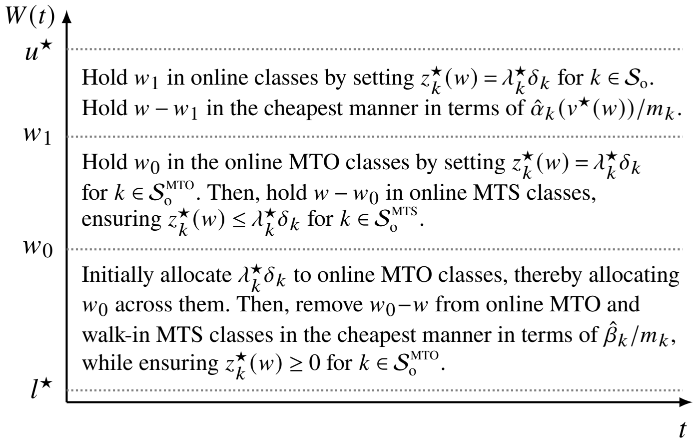


Figure 2 **Illustration of the candidate workload configuration $z^\star(w)$.**

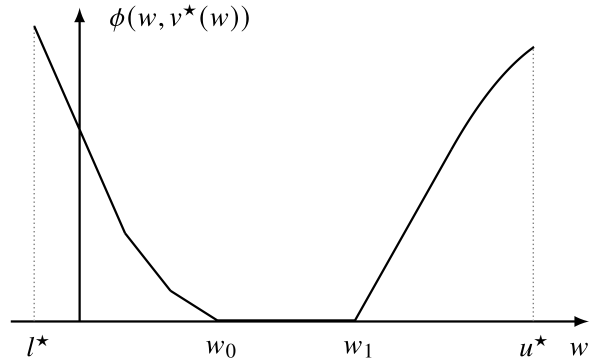


Figure 3 **The effective state cost under the candidate workload configuration $\phi(w, v^\star(w))$.**

walk-in classes. Economically, in this region, the firm asymptotically operates close to the quote times and holds no finished-goods inventory, thereby incurring no state costs.

Low-Congestion Region. For $w \in [l^\star, w_0)$, the workload is insufficient to achieve an effective state cost of zero, and earliness or holding costs are inevitable; see Figure 3. In this region, a workload of w_0 is initially allocated to the online MTO classes (as described above), with an effective state cost of zero. Then, the excess workload $w_0 - w$ is shed from the online MTO and walk-in MTS classes in the cheapest manner in terms of $\hat{\beta}_k/m_k$, which captures the effective earliness and holding cost per unit of work. No workload is held in the online MTS and walk-in MTO classes. Economically, in this region, the firm prefers to incur some earliness or holding costs rather than idle the server, and does so in the least costly manner.

High-Congestion Region. For $w \in (w_1, u^\star]$, the workload is too large to be held entirely in the online classes, and thus tardiness or abandonment costs are inevitable; see Figure 3. In this region, a workload of w_1 is initially allocated to the online classes (as described above), with an effective state cost of zero. The remaining workload $w - w_1$ is then allocated to the cheapest class $k \in \mathcal{S}$ in terms of $\hat{\alpha}_k(v^\star(w))/m_k$, which reflects the aggregate tardiness and abandonment cost per unit of work. Economically, in this region, the firm prefers to incur some tardiness or abandonment costs rather than reject orders, and does so by holding the additional workload in the least costly manner. In this regime alone, abandonments affect the workload distribution (through the value function $v^\star(w)$) by impacting where the excess workload is held.

The following result formalizes the optimality of the constructed candidate policy.

THEOREM 2. *The barrier policy $(L^\star, U^\star, z^\star, \theta^\star)$ with lower barrier at $l^\star$, upper barrier at $u^\star$, effective drift rate function $\theta^\star$ defined by (63), and workload configuration function $z^\star$ defined by (64) is an optimal solution to the EWF (49), and it has a long-run average expected cost of $\gamma^\star$.*

7. Proposed Policy

This section proposes a dynamic control policy for the original control problem introduced in Section 3 by interpreting the solution to the workload formulation in the context of the original control problem. The proposed policy consists of three components: pricing, scheduling, and rejection. We first present the proposed policy with dynamic prices, and then discuss the other two variants, namely, the proposed policy with online-online dynamic prices and static prices. Recall that we consider a sequence of systems indexed by n, whose formal limit is the BCP (37). The original control problem corresponds to a particular n. Thus, to define the proposed policy, we first fix n and use it to unscale the processes of interest.

Theorem 2 establishes that the optimal solution to the workload formulation has the following structure. First, the workload is constrained to $[l^\star, u^\star]$ by imposing a lower barrier at $l^\star$ via the effective idleness process L and an upper barrier at $u^\star$ via the effective rejection process U. Second, for each $w \in [l^\star, u^\star]$, the workload is distributed across classes according to the optimal workload configuration $z^\star(w)$. Third, for each $w \in [l^\star, u^\star]$, the effective drift rate $\theta^\star(w)$ is applied; see (63).

The proposed policy implements this structure in the context of the original system. Since the proposed policy depends on the aggregate workload, we begin by defining the unscaled workload process $W^n = \{W^n(t) : t \geq 0\}$ as $W^n(t) \coloneqq \sum_{k\in\mathcal{S}} m_k Q_k^n(t)$ for $t \geq 0$. Although the workload process in the workload formulation lives in $[l^\star, u^\star]$, the scaled nominal workload process $W^n(t)/\sqrt{n}$ may leave this interval. We therefore define the projected workload as $\mathscr{W}^n(t) \coloneqq l^\star \vee (W^n(t)/\sqrt{n} \wedge u^\star)$ for $t \geq 0$, which projects $W^n(t)/\sqrt{n}$ into $[l^\star, u^\star]$, and express the proposed policy in terms of the projected workload process.

Pricing Policy. The proposed pricing policy employs the following price vector:

$$p^n(t) \coloneqq \Lambda^{-1}(\lambda^\star) - \frac{\nabla\Lambda^{-1}(\lambda^\star)}{\sqrt{n}} \frac{H^{-1}m}{2} \nu^\star(\mathscr{W}^n(t)), \quad t \geq 0, \tag{68}$$

where $\nabla\Lambda^{-1}(\lambda^\star)$ denotes the Jacobian of Λ^{-1} evaluated at $\lambda^\star$; see (17), (19), and Corollary EC.6 in Appendix EC.3. Since $\nu^\star$ is increasing, the proposed pricing policy induces an effective demand rate that decreases with congestion. In particular, when workload is low, prices remain close to the nominal static prices; as workload increases, prices are increased to reduce demand, and hence profit, to mitigate congestion. Because the workload evolves on a slower time scale than individual arrivals and production completions, prices change gradually over time, providing a stable environment for customers while regulating congestion.

Rejection Policy. The solution to the EWF (49) imposes an upper barrier on the workload process at $u^\star$ via the effective rejection process. We interpret this as the policy that rejects orders of product $k^\star$—which has the lowest rejection cost per unit of work—when $\mathscr{W}^n(t)$ reaches the rejection threshold $u^\star$.

Scheduling Policy. The proposed scheduling policy consists of two components: production and reallocation decisions. We introduce small safety stocks, denoted by s_k for $k \in \mathcal{S}$, to account for the difference between the diffusion-scaled queue lengths prescribed by the optimal workload configuration function and the dynamics of the original system; the safety stocks can be calibrated via simulation. The scheduling policy implements a state-dependent target-tracking rule: it treats $\sqrt{n}z^\star(\mathscr{W}^n(t)) + s$ as a moving target for the desired workload distribution and steers the system toward this target. Our policy first applies reallocation decisions and then uses production decisions to correct for any remaining imbalances.

Reallocation Decisions. Reallocation is triggered only when the queue length of walk-in MTS product $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$ falls below its safety stock, and the backlog of one of its associated online MTS classes exceeds its target workload distribution. That is, when $Q_k^n(t) < \min(s_k, 0)$, the proposed policy reallocates finished-goods inventory from class k to the classes $j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)$ for which $Q_j^n(t) > \sqrt{n}z_j^\star(\mathscr{W}^n(t)) + s_j$. Among such eligible online MTS product classes, the policy first prioritizes those for which $Q_j^n(t) > \sqrt{n}\lambda_j^\star\delta_j + s_j$, and then the remaining product classes. Within each group, products are prioritized in descending order of their effective state costs per unit of work, i.e., $\hat{\alpha}_j\big(v^\star(\mathscr{W}^n(t))\big)/m_j$. Structurally, this is an index-type policy: finished-goods inventory is directed to the online classes whose backlog exceeds the target workload distribution in descending order of their effective state cost per unit of work, subject to feasibility constraints.

Production Decisions. The solution to the EWF (49) imposes a lower barrier on the workload process at $l^\star$ via the effective idleness process. We interpret this as a policy that idles when the projected workload reaches the idling threshold $l^\star$ and the queue length is below the desired target, i.e.,

$$\mathscr{W}^n(t) = l^\star \quad \text{and} \quad Q_k^n(t) \le \sqrt{n}z_k^\star(\mathscr{W}^n(t)) + s_k \quad \text{for all } k \in \mathcal{S}.$$

This intentional idleness is used to avoid excessive earliness and holding costs. When the above condition is violated, the server works. First, it aims to align the workload distribution with the desired target. It does so by choosing, among product classes whose queue length is above the desired target, i.e., $Q_k^n(t) > \sqrt{n}z_k^\star(\mathscr{W}^n(t)) + s_k$, the class with the largest change in the effective state cost per unit of work, given by

$$\tilde{v}_k(t) := \begin{cases} \hat{\alpha}_k\big(v^\star(\mathscr{W}^n(t))\big)/m_k, & \text{if } z_k^\star(\mathscr{W}^n(t)) \ge \lambda_k^\star\delta_k, \\ -\hat{\beta}_k/m_k, & \text{otherwise.} \end{cases}$$

If such a product exists, the server produces the corresponding good; for online MTS products, once the good is produced, it is added to the corresponding walk-in buffer.

If all queue lengths are below their desired target, i.e., $Q_k^n(t) \le \sqrt{n}z_k^\star(\mathscr{W}^n(t)) + s_k$ for all $k \in \mathcal{S}$ and $\mathscr{W}^n(t) > l^\star$, the server produces the product with the largest decrease (equivalently, smallest increase) in the effective state cost per unit of work, i.e., class $k^\star(\mathscr{W}^n(t), v^\star(\mathscr{W}^n(t)))$ products; see (EC.37) in Appendix EC.2.2. Specifically, when $\mathscr{W}^n(t) \in (w_1, u^\star]$, the product that results in the largest decrease in the effective state cost is produced; when $\mathscr{W}^n(t) \in [l^\star, w_0)$, the online MTO or walk-in MTS product that results in the smallest increase in the effective state cost is produced; and when $\mathscr{W}^n(t) \in [w_0, w_1]$, the good associated with an online MTS product that has a positive queue length is produced.

Policies with Restricted Pricing Flexibility. The policy described above permits the system manager to dynamically adjust prices for all products. As discussed in Section 3.1, our framework naturally accommodates settings with limited pricing flexibility. We next describe two practically relevant variants, both of which retain the scheduling and rejection structure derived above and differ only in pricing flexibility.

The first variant, referred to as the proposed policy with static prices, employs a fixed price vector. This corresponds to solving the equivalent workload formulation under a constant drift rate and choosing the drift rate that minimizes its long-run average cost. The corresponding static price vector is obtained by inverting the demand function. The second variant, referred to as the proposed policy with online-only dynamic prices, allows dynamic pricing for the online channel while keeping walk-in prices fixed. This corresponds to solving the Brownian control problem under constraints that fix the prices of walk-in products. For a given static price vector for the walk-in products, we solve the workload formulation and choose the static price vector for the walk-in products that minimizes its long-run average cost.

8. Numerical Study

This section illustrates the effectiveness of our proposed policy and provides managerial insights on the dynamic control of omnichannel hybrid production systems. We consider a system with one MTS good and one MTO good, each available through both walk-in and online channels. The online channel operates with a single 30-minute quote time. Thus, there are four total products. We numerically label the products so that $\mathcal{S} = \{1, 2, 3, 4\}$, and order them so that $\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}} = \{1\}$, $\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} = \{2\}$, $\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}} = \{3\}$, and $\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}} = \{4\}$. Activities are numbered as follows: for $k = 1, 2, 3$, activity k corresponds to the production of product k, while activity 4 corresponds to the reallocation of finished MTS goods from the walk-in buffer to fulfill online orders.

We assume each unit of time corresponds to one hour. Customers arrive according to a Poisson process with rate Λ_0, and choose a product according to a nested logit demand model. (This demand model satisfies Assumptions 1 and 4.) In this model, a customer first selects an ordering channel (online or walk-in) and then selects a product within that channel. We set $\Lambda_0 = 666.67$, and set the parameters of the nested logit model to $a_k = -0.99$, $b_k = 1.79$, and $c_i = 1$ for $k \in \mathcal{S}$ and $i \in \{\text{Online}, \text{Walk-in}\}$; see Appendix EC.5 for details.

We set the variable cost of production to $\gamma_k = 0.3$ for $k \in \mathcal{S}$. Under this parameterization, the nominal static prices are $p_k^\star = 1$ for $k \in \mathcal{S}$, which yield demand rates of $\Lambda_k(p^\star) = 33.33$ for $k \in \mathcal{S}$. Under the nominal prices, a customer makes a purchase with probability 0.2 and, conditional on making a purchase, selects each product with equal probability.[6] Furthermore, given these parameters, the total variable cost of production accounts for 30% of the nominal revenue, aligning with industry statistics (TouchBistro 2025).[7]

The system parameters for the base case are presented in Table 1. Under the nominal demand rates, the server is fully utilized and allocates approximately $\rho_k = 1/3$ of its time to each product $k \in \{1, 2, 3\}$. To isolate the effect of abandonment on system performance, the abandonment rates are set to zero in the base

Table 1 System parameters in the base case.

	Product 1	Product 2	Product 3	Product 4
Product characteristics				
Channel	Walk-in	Online	Walk-in	Online
Fulfillment mode	MTO	MTO	MTS	MTS
Quote time, δ_k	–	0.5	–	0.5
Demand and operational parameters				
Nominal static price, $p_k^\star$	1	1	1	1
Demand rate under nominal prices, $\Lambda_k(p^\star)$	33.33	33.33	33.33	33.33
Production rate, μ_k	100	100	200	–
Abandonment rate, ℓ_k	0	–	0	–
Cost parameters				
Variable cost of production, γ_k	0.30	0.30	0.30	0.30
Abandonment cost, d_k	0.80	–	0.80	–
Tardiness cost, α_k	0.05	0.05	0.05	0.05
Earliness cost, β_k	–	0.025	–	–
Holding cost, h_k	–	–	0.0125	–
Rejection cost, r_k	0.80	0.80	0.80	0.80

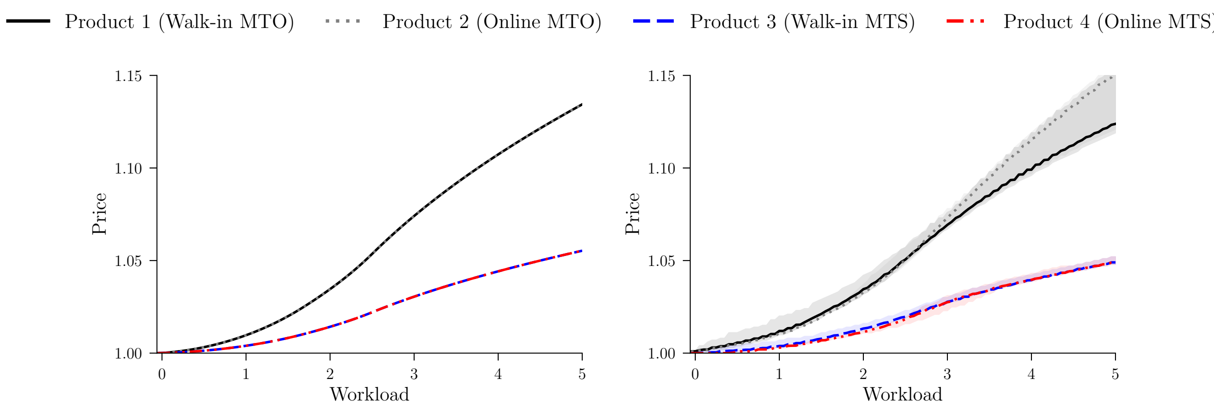
Notes. Entries marked "–" indicate parameters that are not applicable.

case, i.e., $\ell_k = 0$ for $k \in \mathcal{S}_{\text{w}}$. This allows us to establish a baseline for system behavior in the absence of abandonment; we subsequently introduce positive abandonment rates in additional experiments to evaluate their impact. To assess the robustness of our findings and to extract insights, we later conduct a sensitivity analysis with respect to several key parameters. We simulate each policy 100 times for 10^6 time units starting from an empty system. To eliminate transient effects, the initial 20% of each run is discarded.

8.1. Structure of the Proposed Policy

This section discusses the structure of our proposed policy with dynamic prices in the base case and draws several structural insights. We set the system scaling parameter to $n = 100$, yielding nominal demand rates of $\lambda_k^\star = 0.33$ for $k \in \mathcal{S}$. It is straightforward to verify that Assumptions 3–4 are satisfied. In the base case, $w_0 = 1.67$, $w_1 = 2.50$, and $k^\star = 1$, and the solution to the workload formulation imposes a lower barrier at $l^\star = -0.06$ and an upper barrier at $u^\star = 8.16$.

We evaluate the effectiveness of the proposed policy by comparing it against a benchmark. However, there is no exact method for solving the original control problem due to the presence of sojourn times in (13). Nevertheless, a reasonable approximation can be obtained by replacing the sojourn times $w_k(t)$ with $Q_k(t)/\lambda_k^\star$ for $k \in \mathcal{S}$ and $t \geq 0$; see, e.g., Rubino and Ata (2009). This approximation yields a semi-Markov decision process (SMDP) formulation of the control problem; see Appendix EC.5.1 for a detailed description. Thus, we use the optimal solution of this surrogate control problem, henceforth referred to as the SMDP policy with dynamic prices, as our primary benchmark. This comparison allows us to assess whether the proposed policy captures the key operational trade-offs of the problem despite relying on a one-dimensional state representation. From a computational standpoint, the SMDP benchmark is demanding due to the curse



(a) Proposed policy with dynamic prices

(b) SMDP policy with dynamic prices

Figure 4 Prices prescribed by our proposed policy and the SMDP policy with dynamic prices as functions of workload.

Notes. Shaded areas for the SMDP policy represent the 95% interval (2.5th–97.5th percentiles) of prices at each workload.

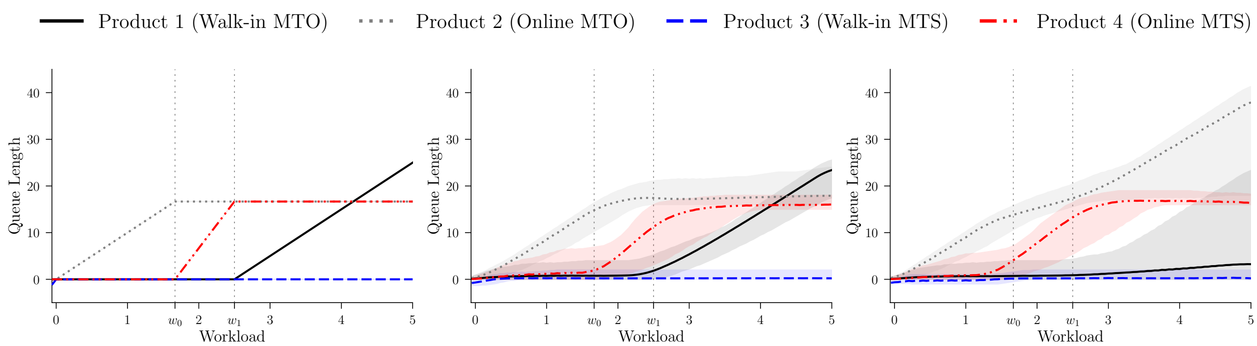
of dimensionality: even with queue lengths truncated at ±75, solving it via policy iteration requires days of computation; see Appendix EC.5.2. In contrast, the workload formulation can be solved almost instantly.

We next describe the pricing, scheduling, and order rejection decisions of our proposed policy and compare them with those of the SMDP policy with dynamic prices. The two policies make strikingly similar decisions, suggesting that the aggregate workload captures most of the economically relevant state information. In Section 8.2, we further show that the two policies achieve similar performance. Under both policies, the workload lies in the interval $[-0.06, 4.85]$ for over 99.9% of the simulation horizon; see Figure EC.1. Accordingly, we focus on this region, which captures the system's typical operating regime.

Pricing Policy. Figure 4a depicts the prices prescribed by the proposed policy at each workload level. Under this policy, prices depend on the aggregate workload through the optimal value function $v^\star(w)$; see (68). Figure 4b depicts the mean and 95% interval of the prices used by the SMDP policy with dynamic prices at each workload level.[8] The prices under the proposed policy closely approximate those of the SMDP benchmark. Specifically, the average price deviation (computed with respect to the stationary distribution of the workload process under the SMDP policy with dynamic prices) is less than 0.08%. Moreover, within the interval $[-0.06, 4.85]$, the prices differ by at most 1.0%, 1.8%, 0.6%, and 0.7% for each product, respectively.

Workload Configuration and Scheduling Policy. Our proposed scheduling policy dynamically engages in production and reallocation activities to steer the system toward the target workload configuration $\sqrt{n}z^\star(w) + s$, where $z^\star(w)$ is the optimal workload configuration and s is the safety stock vector (yielding the best performance); see Section 7. The target workload configuration for the base case is depicted in Figure 5a.

The thresholds $w_0 = 1.67$ and $w_1 = 2.5$ partition the workload space into three congestion regions; see Section 6.3. In the intermediate-congestion region $[w_0, w_1] = [1.67, 2.5]$, the workload is held entirely in the online classes with zero effective state cost. This highlights the role of the online channel as a scheduling buffer that absorbs workload without incurring state costs. Specifically, the policy allocates $w_0 = 1.67$ units



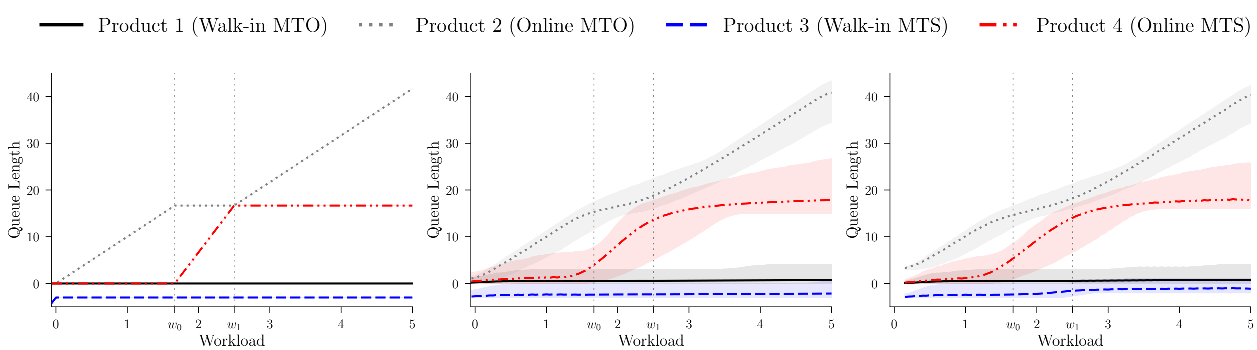
(a) Target workload configuration $\sqrt{n}z^{\star} + s$ with $s = (0, 0, 0, 0)$

(b) Queue lengths under the proposed policy with dynamic prices

(c) Queue lengths under the SMDP policy with dynamic prices

Figure 5 Optimal workload configuration and the queue lengths used under the proposed policy and the SMDP policy with dynamic prices for abandonment rates $\ell_k = 0$ for $k \in \mathcal{S}_{\mathrm{w}}$.

Notes. Shaded areas represent the 95% interval (2.5th-97.5th percentiles) of the queue lengths at each workload level.



(a) Target workload configuration $\sqrt{n}z^{\star} + s$ with $s = (0, 0, -3, 0)$

(b) Queue lengths under the proposed policy with dynamic prices

(c) Queue lengths under the SMDP policy with dynamic prices

Figure 6 Optimal workload configuration and the queue lengths used under the proposed policy and the SMDP policy with dynamic prices for abandonment rates $\ell_k = 1$ for $k \in \mathcal{S}_{\mathrm{w}}$.

Notes. Shaded areas represent the 95% interval (2.5th–97.5th percentiles) of the queue lengths at each workload level.

of workload to the online MTO class (product 2) and the remaining $w - w_0$ units to the online MTS class (product 4), with no workload allocated to the walk-in classes; see Figure 5a.

In the low-congestion region $[l^{\star}, w_0) = [-0.06, 1.67)$, the workload is insufficient to attain zero effective state cost, and earliness or holding costs are unavoidable. Since holding costs exceed earliness costs in the base case, when $w \in [0, w_0)$, the workload is allocated entirely to the online MTO class (product 2), incurring only earliness costs. When $w \in [l^{\star}, 0)$, the workload is instead held entirely as finished-goods inventory in the walk-in MTS class (product 3), incurring holding costs in addition to earliness costs; see Figure 5a. When the workload reaches the lower barrier $l^{\star} = -0.06$, the server idles.

In the high-congestion region $(w_1, u^{\star}] = (2.5, 8.16]$, the workload is too high to attain zero effective state cost, and tardiness or abandonment costs are unavoidable. Thus, w_1 units of workload are allocated to the online classes (products 2 and 4) without incurring earliness or tardiness costs, while the remaining $w - w_1$

units are allocated to the classes with the lowest aggregate tardiness and abandonment cost per unit of work. In the base case, where abandonment rates are zero, the walk-in MTO (product 1) and online MTO (product 2) classes have the lowest tardiness cost per unit of work. Thus, the remaining $w - w_1$ units can be allocated across these classes in any combination; our proposed workload configuration assigns this workload entirely to the walk-in MTO class; see Figure 5a. When abandonment rates are positive, holding workload in the walk-in classes becomes more costly and the remaining workload is instead allocated entirely to the online MTO class (product 2); see Figure 6a. This illustrates a key structural effect of abandonment: as abandonment rates increase, congestion is shifted away from the walk-in classes and toward the online classes.

Figures 5 and 6 compare the target workload configuration and queue lengths generated by the proposed policy and the SMDP policy for abandonment rates of zero and one, respectively. In each figure, panel (a) depicts the target workload configuration $\sqrt{n}z^{\star} + s$, panel (b) depicts the mean queue lengths and the 95% confidence bands at each workload level under the proposed policy, and panel (c) depicts the corresponding quantities under the SMDP policy. We observe that the queue lengths under the proposed policy in panel (b) closely track the target workload configuration in panel (a), indicating that the proposed scheduling policy effectively implements the structure prescribed by the Brownian approximation. Moreover, the queue lengths under the proposed policy and the SMDP policy are strikingly similar, despite the former depending only on the aggregate workload and the latter using the full queue-length vector. This provides strong evidence that the aggregate workload captures most of the economically relevant state information in the system.

Rejection Policy. Rejection is used as a last resort: when workload reaches $u^{\star} = 8.16$, the marginal cost of an additional unit of work exceeds the lowest rejection cost per unit of work, so product 1 orders are rejected.

8.2. Results and Managerial Insights

To examine the performance of our proposed policy and quantify the value of pricing, we compare six policies. The first three are the variants of our proposed policy with dynamic prices, online-only dynamic prices, and static prices as described in Section 7. The fourth policy is the SMDP policy with dynamic prices, as described in Section 8.1. The fifth policy, referred to as the SMDP policy with nominal prices, uses the nominal static price vector $p^{\star}$ and solves the SMDP with this price vector to obtain the rejection and scheduling decisions; see Appendix EC.5.1. The sixth policy solves the SMDP with a static price vector optimized via gradient descent, i.e., for a given price vector, we solve the SMDP and then optimize the price vector to minimize the long-run average cost of the SMDP.

Pricing Behavior Across Dynamic and Static Policies. Table EC.1 reports the mean and 95% interval for the prices used by each policy in the base case. Across all policies and products, prices remain within 13% of the nominal static prices. This indicates that the benefits of dynamic pricing can be achieved with relatively modest price adjustments, consistent with regulatory preferences. Moreover, the proposed policies with static and dynamic prices use prices that are close to those of their SMDP counterparts. The static

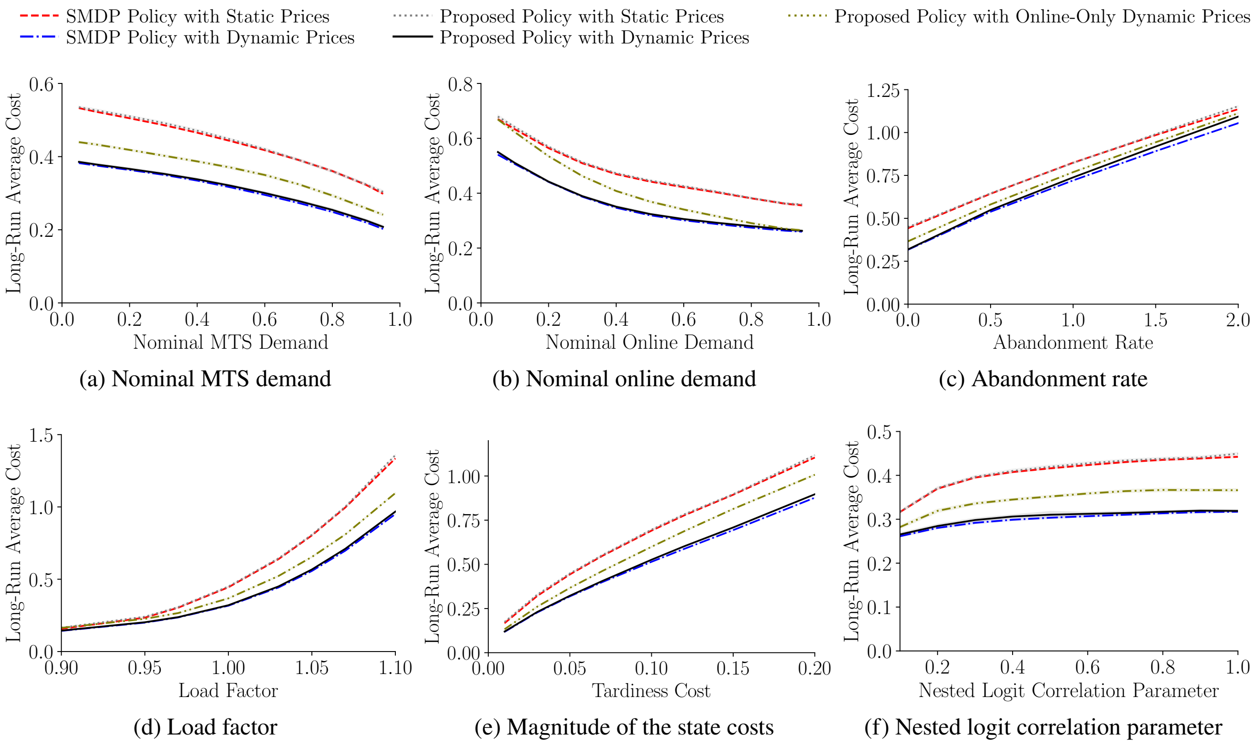


Figure 7 Comparative evaluation of long-run average cost under varying system parameters.

Notes. The nominal MTS (resp., online) demand is the fraction of demand choosing the MTS good (resp., online channel) under the nominal static prices. Shaded areas indicate the 95% confidence band.

pricing policies use prices that are modestly higher (about 1–2%) than the average prices used by the dynamic pricing policies; see Table EC.1. However, dynamic pricing policies exhibit greater dispersion: while the average price is lower, some customers face prices up to 10% higher. This dispersion reflects the ability of dynamic pricing to differentiate across system states, charging higher prices when congestion is high and lower prices otherwise, thereby improving performance without uniformly increasing prices.

Performance Comparison Across Policies. To assess the robustness of our proposed policy and quantify the value of pricing, we conduct a sensitivity analysis across several key model parameters. Figure 7 reports the long-run average cost of each policy across a range of parameter values, while Tables EC.2–EC.7 also report their performance gaps relative to the SMDP policy with dynamic prices.

We draw several key conclusions. First, our proposed policy with dynamic prices performs well, as evidenced by its consistently small gap relative to the SMDP policy with dynamic prices, with an average gap of 1.6% across all instances (and never exceeding 4%). Second, our proposed policy with static prices is near-optimal within the class of static pricing policies, as evidenced by its small gap relative to the SMDP policy with static prices, with an average gap of 1.4% across all instances (and never exceeding 5%). Together, these results show that the Brownian approximation captures the key operational trade-offs governing both pricing and scheduling decisions in omnichannel hybrid production systems.

The strong performance of our proposed policy with static prices is particularly noteworthy, as improving scheduling is a central operational priority for many omnichannel firms; see Section 1. But this requires carefully calibrating the static prices and dynamically coordinating production across products, channels, and quote times, which are precisely the decisions our proposed policy with static prices addresses.

Third, pricing contributes materially to performance. Using optimized static prices that account for congestion delivers significant performance gains (of over 70% in the base case) relative to the nominal prices; see Table EC.2. For firms with dynamic pricing capabilities, further gains (of over 27% in the base case) are achievable relative to static pricing. Finally, our comparative analysis also yields broader insights into when dynamic pricing delivers the greatest value, which we discuss next.

Demand Composition. First, we examine the impact of demand composition, namely, the fraction of customers choosing the MTS good and the fraction choosing the online channel. We vary the demand function parameters to change the nominal MTS demand, i.e., $\sum_{k\in\mathcal{S}^{\text{MTS}}}\lambda_k^\star$, and the nominal online demand, i.e., $\sum_{k\in\mathcal{S}_o}\lambda_k^\star$, while keeping the total demand rate and the nominal static prices fixed, thereby keeping the nominal profit rate fixed. Figures 7a and 7b show that increasing the share of either MTS or online demand reduces costs across all policies due to increased flexibility. Specifically, MTS goods provide an inventory buffer, while online products provide a scheduling buffer; see Sections 6.3 and 8.1.

Moreover, the value of dynamic pricing (relative to static pricing) increases with both the nominal MTS demand and the nominal online demand; see Tables EC.2 and EC.3. This is because, as either quantity increases, states with high congestion costs are visited less frequently. Thus, dynamic pricing can keep prices relatively low in most states and increase them only when congestion rises.

These findings have several managerial implications. First, firms with substantial online demand, or the ability to shift demand online, stand to benefit most from dynamic pricing. A similar, albeit more modest, conclusion applies to firms with high MTS demand. Second, as the nominal online demand increases, the gap between the proposed policy with dynamic prices and the proposed policy with online-only dynamic prices shrinks. This suggests that dynamically pricing only the online channel can be a suitable entry point for adopting dynamic pricing, particularly for firms with a strong online channel. This is especially relevant in practice, as online channels offer cleaner demand data and simpler implementation. This approach enables firms to realize significant gains by initially focusing on the online channel while gradually developing the capabilities needed to dynamically price the walk-in channel.

Abandonment Rates. Second, we examine the impact of customer abandonments. To that end, we vary the abandonment rates ℓ_1 and ℓ_3, while maintaining $\ell_1 = \ell_3$. Figure 7c shows that as abandonment rates increase, the costs across all policies increase (by over 150% as abandonment rates rise from zero to two), while the relative value of dynamic pricing decreases; see Table EC.4. This pattern is intuitive: abandonments partially relieve congestion, thereby substituting for one of the primary roles of dynamic pricing, namely, regulating demand. However, abandonments do so at high cost, whereas dynamic pricing optimally shapes demand. Moreover, although relative gaps narrow, when abandonment rates are moderate, the gains from dynamic pricing remain substantial (over 14% at an abandonment rate of one).

Load Factor. Third, we examine the impact of the load factor $\sum_{k\in\mathcal{S}} \lambda_k^\star/\mu_k$ by varying the rate at which customers arrive Λ_0. Figure 7d shows that as the load factor increases, the costs across all policies and the value of dynamic pricing relative to static pricing increase; see Table EC.5. As the load factor increases, so does congestion, increasing the value of policies that can dynamically adjust the demand rate. This suggests that firms operating near capacity stand to gain the most from dynamic pricing.

State Costs. Fourth, we examine the impact of the state (earliness, tardiness, and holding) costs by varying the magnitude of the tardiness costs α_k proportionally while holding the ratios β_k/α_k and h_k/α_k fixed. As depicted in Figure 7e, increasing the state costs increases the costs across all policies. Moreover, dynamic pricing delivers consistently large gains (over 25% across all parameter values), reinforcing its appeal as a broadly applicable strategy; see Table EC.6.

Nested Logit Correlation Parameter. Finally, we examine the impact of the correlation parameter in the nested logit model; see Appendix EC.5.3. Figure 7f shows that the value of dynamic pricing (relative to static pricing) decreases as the correlation parameter decreases; see Table EC.7. As the correlation parameter decreases, products within a channel (nest) become closer substitutes, so price adjustments increasingly redistribute demand across products rather than reduce effective demand, limiting the ability of dynamic pricing to control congestion and reducing its advantage over static pricing.

Notes

[1]See Panera Bread (2022), Sweetgreen (2024), Soper (2024), and Chipotle (2026) for these figures.

[2]If no MTS good is offered or if some MTS goods are available only through the online channel, the analysis would introduce notational complexities without yielding additional insights.

[3]The units of production and reallocation activities differ: for $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$, T_k represents the cumulative units of reallocation undertaken; for $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$, T_k represents the total server time allocated to production.

[4]The positive and negative parts of $x \in \mathbb{R}$ are $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$, respectively.

[5]MTS goods are stored in walk-in buffers and reallocated to satisfy online demand; see Section 3.2. Including variable production costs for online MTS goods is a modeling convention for consistency.

[6]Several reports cite online conversion rates of 15–25% (Korniichuk and Boryczka 2021).

[7]Fixed costs typically account for approximately 60% of the revenue (TouchBistro 2025), corresponding to a fixed production cost of 80 per unit of time in our example.

[8]Many queue length configurations share the same aggregate workload, and the plotted quantity varies across them; the 95% interval reports the 2.5th to 97.5th percentile at each workload level.

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EC.1. Formal Derivation of the Approximating Brownian System

This section presents a formal derivation of the approximating Brownian control problem discussed in Section 4. The analysis given below does not constitute a rigorous convergence proof of the pre-limit production system to its Brownian approximation. However, the arguments made in support of the Brownian approximation can be viewed as a broad outline of such a proof.

There are two common approaches used to formally derive approximating Brownian systems: weak convergence arguments and functional strong approximations. The weak convergence approach involves arguing that the distribution of a sequence of centered and scaled stochastic processes converges to the distribution of a limiting process; see, e.g., Harrison (1988, 2003). On the other hand, the functional strong approximation approach involves arguing the almost sure convergence of a sequence of stochastic processes to a limiting process; see, e.g., Çelik and Maglaras (2008) and Ata and Tongarlak (2013). Although both methods produce the same approximating Brownian system, we opt for a weak convergence argument.

The general procedure is as follows: First, we consider a sequence of systems indexed by n, operating under conditions of heavy traffic. Second, we center and scale the processes of interest. Finally, we let n get large and replace these processes with their formal limits; see Harrison (1988, 2003) for a more detailed explanation of this procedure. Before proceeding with the analysis, we impose two technical assumptions that underlie the mathematical development. First, all random elements are defined on a fixed probability space. Second, all continuous-time stochastic processes have sample paths that are right-continuous with finite left limits (RCLL).

Fluid and Diffusion Scaled Processes. We begin by defining the fluid-scaled and diffusion-scaled processes. For $k \in \mathcal{S}$, the fluid-scaled instantaneous demand rate process is defined as

$$\bar{\lambda}_k^n(t) := \frac{\lambda_k^n(t)}{n} = \lambda_k^\star + \frac{\zeta_k(t)}{\sqrt{n}}, \quad t \geq 0. \tag{EC.1}$$

For $k \in \mathcal{S}$, the fluid-scaled queue length process is defined as

$$\bar{Q}_k^n(t) := \frac{Q_k^n(t)}{n} = \frac{Z_k^n(t)}{\sqrt{n}}, \quad t \geq 0. \tag{EC.2}$$

For $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$, the diffusion-scaled production process is defined as

$$\hat{S}_k^n(t) := \frac{S_k^n(t) - \mu_k^n t}{\sqrt{n}}, \quad t \geq 0. \tag{EC.3}$$

For $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$, the scaled (deterministic) reallocation process is defined as

$$\hat{S}_k^n(t) := \frac{\lfloor \mu_k^n t \rfloor - \mu_k^n t}{\sqrt{n}}, \quad t \geq 0. \tag{EC.4}$$

For $k \in \mathcal{S}$, the diffusion-scaled Poisson process for order arrivals is defined as

$$\hat{N}_k^n(t) := \frac{N_k(nt) - nt}{\sqrt{n}}, \quad t \ge 0. \tag{EC.5}$$

For $k \in \mathcal{S}_{\mathrm{w}}$, the diffusion-scaled Poisson process for customer abandonments is defined as

$$\hat{M}_k^n(t) := \frac{M_k(nt) - nt}{\sqrt{n}}, \quad t \ge 0. \tag{EC.6}$$

Note from the development in Section 3.2 that $S_k = \{S_k(t) : t \ge 0\}$ is a renewal process with rate μ_k for each $k \in \mathcal{S}$. It follows from the functional central limit theorem for renewal processes and the independence of the processes S, N, and M that the diffusion-scaled processes $\hat{S}^n$, $\hat{N}^n$, and $\hat{M}^n$ converge in distribution to independent multidimensional Brownian motions (of appropriate dimension) with independent components; see Billingsley (1999, Section 14) for details.

Scaled Queue Length Process for the MTO Products. We next rewrite the scaled queue length processes for the MTO products: For $k \in \mathcal{S}^{\mathrm{MTO}}$ and $t \ge 0$, we have that

$$\begin{aligned}
Q_k^n(t) &= N_k\Big(\int_0^t \lambda_k^n(s)\,ds\Big) - S_k^n\big(T_k^n(t)\big) - M_k\Big(\int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k^n\big[Q_k^n(s)\big]^+ ds\Big) - R_k^n(t)\\
&= N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) + \Big[n\int_0^t \bar{\lambda}_k^n(s)\,ds - n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big]\\
&\quad - S_k^n\big(T_k^n(t)\big) + \Big[\mu_k^n T_k^n(t) - \mu_k^n T_k^n(t)\Big] - M_k\Big(\int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big)\\
&\quad + \Big[\int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds - \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big] - R_k^n(t)\\
&= \Big[N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) - n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] + n\int_0^t \bar{\lambda}_k^n(s)\,ds - \Big[S_k^n\big(T_k^n(t)\big) - \mu_k^n T_k^n(t)\Big] - \mu_k^n T_k^n(t)\\
&\quad - \Big[M_k\Big(\int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big) - \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big]\\
&\quad - \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds - R_k^n(t),
\end{aligned} \tag{EC.7}$$

where the first equality follows from (3), the second equality from (26) and (EC.1), and the third equality by the associative property. Dividing both sides of (EC.7) by $\sqrt{n}$ and using (20) gives

$$\begin{aligned}
Z_k^n(t) &= \frac{1}{\sqrt{n}}\Big[N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) - n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] + \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds - \frac{1}{\sqrt{n}}\Big[S_k^n\big(T_k^n(t)\big) - \mu_k^n T_k^n(t)\Big]\\
&\quad - \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) - \frac{1}{\sqrt{n}}\Big[M_k\Big(\int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big) - \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big]\\
&\quad - \frac{1}{\sqrt{n}}\int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds - \frac{1}{\sqrt{n}}R_k^n(t)\\
&= \hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) + \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds - \hat{S}_k^n\big(T_k^n(t)\big) - \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) + \frac{1}{\sqrt{n}}\Big[\mu_k^n\rho_k t - \mu_k^n\rho_k t\Big]
\end{aligned}$$

$$- \hat{M}_k^n\Big(\int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big) - \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Z_k^n(s)\big]^+ds - O_k^n(t), \tag{EC.8}$$

where the second equality follows from (20), (EC.2)–(EC.3), and (EC.5). Then, rearranging the terms on the right-hand side of (EC.8) gives

$$\begin{aligned}
Z_k^n(t) &= \Big[\hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) - \hat{S}_k^n\big(T_k^n(t)\big) - \hat{M}_k^n\Big(\int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big)\Big]\\
&\quad + \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds - \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Z_k^n(s)\big]^+ds - \frac{1}{\sqrt{n}}\mu_k^n\rho_k t\\
&\quad - O_k^n(t) - \frac{1}{\sqrt{n}}\mu_k^n\Big[T_k^n(t) - \rho_k t\Big]\\
&= \Big[\hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) - \hat{S}_k^n\big(T_k^n(t)\big) - \hat{M}_k^n\Big(\int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big) - \eta_k\rho_k t\Big]\\
&\quad + \sqrt{n}\big(\lambda_k^\star - \mu_k\rho_k\big)t + \int_0^t \zeta_k(s)\,ds - \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Z_k^n(s)\big]^+ds\\
&\quad - O_k^n(t) + \Big(\mu_k + \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t),
\end{aligned} \tag{EC.9}$$

where the second equality follows from (22), (EC.1), and rearranging terms. Finally, we define the process $\{X_k^n(t) : t \geq 0\}$ for $k \in \mathcal{S}^{\mathrm{MTO}}$ as follows:

$$X_k^n(t) = \hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) - \hat{S}_k^n\big(T_k^n(t)\big) - \hat{M}_k^n\Big(\int_0^t \ell_k\mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\big[\bar{Q}_k^n(s)\big]^+ds\Big) - \eta_k\rho_k t. \tag{EC.10}$$

Using the definition of ρ_k in (18) and substituting (EC.10) into (EC.9), we obtain the following expression for the scaled queue length process for the MTO products:

$$Z_k^n(t) = X_k^n(t) + \int_0^t \zeta_k(s)\,ds - \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Z_k^n(s)\big]^+ds - O_k^n(t) + \Big(\mu_k + \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t). \tag{EC.11}$$

Scaled Queue Length Process for the Walk-In MTS Products. We next rewrite the scaled queue length processes for the walk-in MTS products: For $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$ and $t \geq 0$, we have that

$$\begin{aligned}
Q_k^n(t) &= N_k\Big(\int_0^t \lambda_k^n(s)\,ds\Big) - S_k^n\big(T_k^n(t)\big) - M_k\Big(\int_0^t \ell_k^n\big[Q_k^n(s)\big]^+ds\Big) - R_k^n(t) + \sum_{j\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} S_j^n\big(T_j^n(t)\big)\\
&= N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) + \Big[n\int_0^t \bar{\lambda}_k^n(s)\,ds - n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] - S_k^n\big(T_k^n(t)\big) + \Big[\mu_k^nT_k^n(t) - \mu_k^nT_k^n(t)\Big]\\
&\quad - M_k\Big(\int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big) + \Big[\int_0^t \ell_k\big[Q_k^n(s)\big]^+ds - \int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big] - R_k^n(t)\\
&\quad + \sum_{j\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} S_j^n\big(T_j^n(t)\big) + \sum_{j\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \Big[\mu_j^nT_j^n(t) - \mu_j^nT_j^n(t)\Big]\\
&= \Big[N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) - n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] + n\int_0^t \bar{\lambda}_k^n(s)\,ds - \Big[S_k^n\big(T_k^n(t)\big) - \mu_k^nT_k^n(t)\Big] - \mu_k^nT_k^n(t)\\
&\quad - \Big[M_k\Big(\int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big) - \int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big] - \int_0^t \ell_k\big[Q_k^n(s)\big]^+ds - R_k^n(t),
\end{aligned}$$

$$+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\Big[S_j^n\big(T_j^n(t)\big)-\mu_j^nT_j^n(t)\Big]+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\mu_j^nT_j^n(t), \tag{EC.12}$$

where the first equality follows from (4), the second equality from (26) and (EC.1), and the third equality by the associative property. Dividing both sides of (EC.12) by $\sqrt{n}$ and using (20) gives

$$\begin{aligned}
Z_k^n(t) &= \frac{1}{\sqrt{n}}\Big[N_k\Big(n\int_0^t\bar{\lambda}_k^n(s)\,ds\Big)-n\int_0^t\bar{\lambda}_k^n(s)\,ds\Big]+\sqrt{n}\int_0^t\bar{\lambda}_k^n(s)\,ds-\frac{1}{\sqrt{n}}\Big[S_k^n\big(T_k^n(t)\big)-\mu_k^nT_k^n(t)\Big]\\
&\quad-\frac{1}{\sqrt{n}}\mu_k^nT_k^n(t)-\frac{1}{\sqrt{n}}\Big[M_k\Big(\int_0^t\ell_k\big[Q_k^n(s)\big]^+ds\Big)-\int_0^t\ell_k\big[Q_k^n(s)\big]^+ds\Big]-\frac{1}{\sqrt{n}}\int_0^t\ell_k\big[Q_k^n(s)\big]^+ds\\
&\quad-\frac{1}{\sqrt{n}}R_k^n(t)+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\frac{1}{\sqrt{n}}\Big[S_j^n\big(T_j^n(t)\big)-\mu_j^nT_j^n(t)\Big]+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\frac{1}{\sqrt{n}}\mu_j^nT_j^n(t)\\
&=\hat{N}_k^n\Big(\int_0^t\bar{\lambda}_k^n(s)\,ds\Big)+\sqrt{n}\int_0^t\bar{\lambda}_k^n(s)\,ds-\hat{S}_k^n\big(T_k^n(t)\big)-\frac{1}{\sqrt{n}}\mu_k^nT_k^n(t)+\frac{1}{\sqrt{n}}\Big[\mu_k^n\rho_kt-\mu_k^n\rho_kt\Big]\\
&\quad-\hat{M}_k^n\Big(\int_0^t\ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big)-\int_0^t\ell_k\big[Z_k^n(s)\big]^+ds-O_k^n(t)+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\hat{S}_j^n\big(T_j^n(t)\big)\\
&\quad+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\frac{1}{\sqrt{n}}\mu_j^nT_j^n(t)+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\frac{1}{\sqrt{n}}\Big[\mu_j^n\lambda_j^\star\mu_j^{-1}t-\mu_j^n\lambda_j^\star\mu_j^{-1}t\Big],
\end{aligned} \tag{EC.13}$$

where the second equality follows from (20) and (EC.2)–(EC.5). Then, rearranging the terms on the right-hand side of (EC.13) gives

$$\begin{aligned}
Z_k^n(t) &= \Big[\hat{N}_k^n\Big(\int_0^t\bar{\lambda}_k^n(s)\,ds\Big)-\hat{S}_k^n\big(T_k^n(t)\big)-\hat{M}_k^n\Big(\int_0^t\ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big)+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\hat{S}_j^n\big(T_j^n(t)\big)\Big]\\
&\quad+\sqrt{n}\int_0^t\bar{\lambda}_k^n(s)\,ds-\int_0^t\ell_k\big[Z_k^n(s)\big]^+ds-\frac{1}{\sqrt{n}}\mu_k^n\rho_kt+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\frac{1}{\sqrt{n}}\mu_j^n\lambda_j^\star\mu_j^{-1}t-O_k^n(t)\\
&\quad-\frac{1}{\sqrt{n}}\mu_k^n\Big[T_k^n(t)-\rho_kt\Big]+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\frac{1}{\sqrt{n}}\mu_j^n\Big[T_j^n(t)-\lambda_j^\star\mu_j^{-1}t\Big]\\
&=\Bigg[\hat{N}_k^n\Big(\int_0^t\bar{\lambda}_k^n(s)\,ds\Big)-\hat{S}_k^n\big(T_k^n(t)\big)-\hat{M}_k^n\Big(\int_0^t\ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big)+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\hat{S}_j^n\big(T_j^n(t)\big)\\
&\quad+\Big(-\eta_k\rho_k+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\eta_j\lambda_j^\star\mu_j^{-1}\Big)t\Bigg]+\sqrt{n}\Big(\lambda_k^\star+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\mu_j\lambda_j^\star\mu_j^{-1}-\mu_k\rho_k\Big)t+\int_0^t\zeta_k(s)\,ds\\
&\quad-\int_0^t\ell_k\big[Z_k^n(s)\big]^+ds-O_k^n(t)+\Big(\mu_k+\frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t)-\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\Big(\mu_j+\frac{1}{\sqrt{n}}\eta_j\Big)Y_j^n(t),
\end{aligned} \tag{EC.14}$$

where the second equality follows from (22), (EC.1), and rearranging terms. Finally, we define the process $X_k^n=\{X_k^n(t):t\geq 0\}$ for $k\in\mathcal{S}_{\text{w}}^{\text{MTS}}$ as follows:

$$\begin{aligned}
X_k^n(t) &= \hat{N}_k^n\Big(\int_0^t\bar{\lambda}_k^n(s)\,ds\Big)-\hat{S}_k^n\big(T_k^n(t)\big)-\hat{M}_k^n\Big(\int_0^t\ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big)\\
&\quad+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\hat{S}_j^n\big(T_j^n(t)\big)-\Big(\eta_k\rho_k-\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)}\eta_j\lambda_j^\star\mu_j^{-1}\Big)t.
\end{aligned} \tag{EC.15}$$

Using the definition of ρ_k in (18) and substituting (EC.15) into (EC.14), we obtain the following expression for the scaled queue length process for the walk-in MTS products:

$$\begin{aligned} Z_k^n(t) &= X_k^n(t) + \int_0^t \zeta_k(s)\,ds - \int_0^t \ell_k\big[Z_k^n(s)\big]^+ ds - O_k^n(t) \\ &\quad + \Big(\mu_k + \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t) - \sum_{j\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)}\Big(\mu_j + \frac{1}{\sqrt{n}}\eta_j\Big)Y_j^n(t). \end{aligned} \tag{EC.16}$$

Scaled Queue Length Process for the Online MTS Products. We next rewrite the scaled queue length processes for the online MTS products: For $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$ and $t \geq 0$, we have that

$$\begin{aligned} Q_k^n(t) &= N_k\Big(\int_0^t \lambda_k^n(s)\,ds\Big) - S_k^n\big(T_k^n(t)\big) - R_k^n(t) \\ &= N_k\Big(n\int_0^t \bar\lambda_k^n(s)\,ds\Big) + \Big[n\int_0^t \bar\lambda_k^n(s)\,ds - n\int_0^t \bar\lambda_k^n(s)\,ds\Big] - S_k^n\big(T_k^n(t)\big) \\ &\quad + \Big[\mu_k^n T_k^n(t) - \mu_k^n T_k^n(t)\Big] - R_k^n(t) \\ &= \Big[N_k\Big(n\int_0^t \bar\lambda_k^n(s)\,ds\Big) - n\int_0^t \bar\lambda_k^n(s)\,ds\Big] + n\int_0^t \bar\lambda_k^n(s)\,ds \\ &\quad - \Big[S_k^n\big(T_k^n(t)\big) - \mu_k^n T_k^n(t)\Big] - \mu_k^n T_k^n(t) - R_k^n(t), \end{aligned} \tag{EC.17}$$

where the first equality follows from (5), the second equality from (EC.1), and the third equality by the associative property. Dividing both sides of (EC.17) by $\sqrt{n}$ and using (20) gives

$$\begin{aligned} Z_k^n(t) &= \frac{1}{\sqrt{n}}\Big[N_k\Big(n\int_0^t \bar\lambda_k^n(s)\,ds\Big) - n\int_0^t \bar\lambda_k^n(s)\,ds\Big] + \sqrt{n}\int_0^t \bar\lambda_k^n(s)\,ds \\ &\quad - \frac{1}{\sqrt{n}}\Big[S_k^n\big(T_k^n(t)\big) - \mu_k^n T_k^n(t)\Big] - \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) - \frac{1}{\sqrt{n}}R_k^n(t) \\ &= \hat N_k^n\Big(\int_0^t \bar\lambda_k^n(s)\,ds\Big) + \sqrt{n}\int_0^t \bar\lambda_k^n(s)\,ds - \hat S_k^n\big(T_k^n(t)\big) - \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) \\ &\quad + \frac{1}{\sqrt{n}}\Big[\mu_k^n\lambda_k^\star\mu_k^{-1}t - \mu_k^n\lambda_k^\star\mu_k^{-1}t\Big] - O_k^n(t), \end{aligned} \tag{EC.18}$$

where the second equality follows from (20) and (EC.4)–(EC.5). Then, rearranging the terms on the right-hand side of (EC.18) gives

$$\begin{aligned} Z_k^n(t) &= \Big[\hat N_k^n\Big(\int_0^t \bar\lambda_k^n(s)\,ds\Big) - \hat S_k^n\big(T_k^n(t)\big)\Big] + \sqrt{n}\int_0^t \bar\lambda_k^n(s)\,ds - \frac{1}{\sqrt{n}}\mu_k^n\lambda_k^\star\mu_k^{-1}t \\ &\quad - O_k^n(t) - \frac{1}{\sqrt{n}}\mu_k^n\Big[T_k^n(t) - \lambda_k^\star\mu_k^{-1}t\Big] \\ &= \Big[\hat N_k^n\Big(\int_0^t \bar\lambda_k^n(s)\,ds\Big) - \hat S_k^n\big(T_k^n(t)\big) - \eta_k\lambda_k^\star\mu_k^{-1}t\Big] + \sqrt{n}\big(\lambda_k^\star - \mu_k\lambda_k^\star\mu_k^{-1}\big)t + \int_0^t \zeta_k(s)\,ds \\ &\quad - O_k^n(t) + \Big(\mu_k + \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t), \end{aligned} \tag{EC.19}$$

where the second equality follows from (22), (EC.1), and rearranging terms. Finally, we define the process $\{X_k^n(t) : t \geq 0\}$ for $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$ as follows:

$$X_k^n(t) = \hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) - \hat{S}_k^n\big(T_k^n(t)\big) - \eta_k \lambda_k^\star \mu_k^{-1} t. \tag{EC.20}$$

By canceling terms and substituting (EC.20) into (EC.19), we obtain the following expression for the scaled queue length process for the online MTS products:

$$Z_k^n(t) = X_k^n(t) + \int_0^t \zeta_k(s)\,ds - O_k^n(t) + \Big(\mu_k + \frac{1}{\sqrt{n}}\eta_k\Big) Y_k^n(t). \tag{EC.21}$$

Cumulative Cost Process. We now derive an expression for the cumulative cost process. Recall that the cumulative profit process $V^n = \{V^n(t) : t \geq 0\}$ for the nth system is given by

$$\begin{aligned} V^n(t) = &\int_0^t \Pi^n\big(\lambda^n(s)\big)\,ds - \sum_{k\in\mathcal{S}} \int_0^t \upsilon_k^n\big(w_k^n(s) - \delta_k^n\big)\,dA_k^n(s) - \sum_{k\in\mathcal{S}_{\text{w}}^{\text{MTS}}} \int_0^t h_k^n\big[Q_k^n(s)\big]^- ds \\ &- \sum_{k\in\mathcal{S}_{\text{w}}} d_k^n M_k\Big(\int_0^t \ell_k^n\big[Q_k^n(s)\big]^+ ds\Big) - \sum_{k\in\mathcal{S}} r_k^n R_k^n(t), \end{aligned} \tag{EC.22}$$

where the process $A_k^n = \{A_k^n(t) : t \geq 0\}$ tracks the number of admitted class $k \in \mathcal{S}$ orders over time in the nth system, and it is given by

$$A_k^n(t) = N_k\Big(\int_0^t \lambda_k^n(s)\,ds\Big) - R_k^n(t). \tag{EC.23}$$

We rewrite (EC.22) by considering each term on the right-hand side individually. For the first term on the right-hand side of (EC.22), applying (17) and (19) gives

$$\Pi^n\big(\lambda^n(s)\big) = \Pi^n\big(n\lambda^\star + \sqrt{n}\zeta(s)\big) = n\Pi\Big(\lambda^\star + \frac{1}{\sqrt{n}}\zeta(s)\Big). \tag{EC.24}$$

Since $\lambda^\star \in \text{interior}(\mathcal{L})$ and Π is twice continuously differentiable on $\mathcal{L}$ (see Assumptions 3 and 4), for sufficiently large n, the multivariate version of Taylor's theorem yields

$$\begin{aligned} \Pi\Big(\lambda^\star + \frac{1}{\sqrt{n}}\zeta(s)\Big) &= \Pi(\lambda^\star) + \frac{1}{\sqrt{n}}\nabla\Pi(\lambda^\star)'\zeta(s) + \frac{1}{2n}\zeta(s)'\,\nabla^2\Pi(\lambda^\star)\,\zeta(s) + o(1/n) \\ &= \Pi(\lambda^\star) + \frac{1}{2n}\zeta(s)'\,\nabla^2\Pi(\lambda^\star)\,\zeta(s) + o(1/n), \end{aligned} \tag{EC.25}$$

where the second equality follows from $\nabla\Pi(\lambda^\star) = 0$ (see Assumption 3). Substituting (EC.25) into (EC.24) and integrating both sides then gives

$$\int_0^t \Pi^n\big(\lambda^n(s)\big)\,ds = n\Pi(\lambda^\star)t + \frac{1}{2}\int_0^t \zeta(s)'\,\nabla^2\Pi(\lambda^\star)\,\zeta(s)\,ds + o(1). \tag{EC.26}$$

For the second term on the right-hand side of (EC.22), letting $\bar{A}_k^n(t) := n^{-1}A_k^n(t)$ for $t \ge 0$ and applying (24) gives

$$\int_0^t \upsilon_k^n\big(w_k^n(s)-\delta_k^n\big)\,dA_k^n(s) = \frac{1}{\sqrt{n}}\int_0^t \upsilon_k\big(w_k^n(s)-\delta_k^n\big)\,dA_k^n(s) = \int_0^t \upsilon_k\big(\sqrt{n}w_k^n(s)-\delta_k\big)\,d\bar{A}_k^n(s),$$

where the second equality follows from (25). Finally, for the fourth term on the right-hand side of (EC.22), applying (20), (23), (26), and (EC.6) gives

$$\sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k^n M_k\Big(\int_0^t \ell_k^n\big[Q_k^n(s)\big]^+ds\Big) = \sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k\Big[\hat{M}_k^n\Big(\int_0^t \ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big) + \int_0^t \ell_k\big[Z_k^n(s)\big]^+ds\Big]. \tag{EC.27}$$

Then, by combining (23)–(27), (EC.22), and (EC.26)–(EC.27), we obtain the following expression for the cumulative cost process:

$$\begin{aligned}\xi^n(t) = &-\frac{1}{2}\int_0^t \zeta(s)'\nabla^2\Pi(\lambda^\star)\zeta(s)\,ds + \sum_{k\in\mathcal{S}}\int_0^t \upsilon_k\big(\sqrt{n}w_k^n(s)-\delta_k\big)\,d\bar{A}_k^n(s) + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}}\int_0^t h_k\big[Z_k^n(s)\big]^-ds\\ &+\sum_{k\in\mathcal{S}_{\mathrm{w}}}\int_0^t d_k\ell_k\big[Z_k^n(s)\big]^+ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k\hat{M}_k^n\Big(\int_0^t \ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big) + \sum_{k\in\mathcal{S}} r_kO_k^n(t) + o(1).\end{aligned} \tag{EC.28}$$

Approximating Brownian Control Problem. We established above that the scaled queue length processes for the MTO, walk-in MTS, and online MTS products satisfy (EC.10)–(EC.11), (EC.15)–(EC.16), and (EC.20)–(EC.21), respectively, and that the cumulative cost process satisfies (EC.28). We now provide a formal limiting argument to justify approximating these processes with their diffusion limits as n gets large. As articulated in Section 4.1—in particular, see (21)—a key assertion in this approximation is that as n grows large, the only allocation policies worthy of consideration are those satisfying

$$T_k^n(t)\approx\rho_k t,\quad k\in\mathcal{S}^{\mathrm{MTO}}\cup\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}\qquad\text{and}\qquad T_k^n(t)\approx\lambda_k^\star\mu_k^{-1}t,\quad k\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}.$$

We now argue that the K-dimensional process $X^n=(X_k^n)$, given by (EC.10), (EC.15), and (EC.20), can be approximated by a K-dimensional Brownian motion X. By (EC.1), observe that $\bar{\lambda}^n\approx\lambda^\star$ as n gets large. Applying the functional central limit theorem for renewal processes and (a loose application of) the random time change theorem, we approximate the first terms on the right-hand side of (EC.10), (EC.15), and (EC.20) by $B_k(\lambda_k^\star t)$ for $k\in\mathcal{S}$, where B_k is a standard Brownian motion; see, e.g., Billingsley (1999). By (21), a similar reasoning applies to the second terms on the right-hand side of (EC.10) and (EC.15) to argue that as n gets large,

$$\hat{S}_k^n(T_k^n(t))\approx\begin{cases}\widetilde{B}_k\big(\lambda_k^\star(1+c_{sk}^2)t\big), & k\in\mathcal{S}^{\mathrm{MTO}},\\ \widetilde{B}_k\big((\lambda_k^\star+c_{sk}^2\mu_k\rho_k)t\big), & k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}},\end{cases}$$

where $\widetilde{B}_k$ is a standard Brownian motion. Moreover, since the queue length processes are expected to be of order $\sqrt{n}$ in the heavy-traffic regime, it follows that $\bar{Q}_k^n(t)\approx 0$ for $k\in\mathcal{S}$ as n gets large; see (EC.2).

Finally, by (EC.4), it is immediate that $\hat{S}_k^n(T_k^n(t))$ for $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$ converges to zero almost surely as n gets large. We conclude that the third term on the right-hand side of (EC.10), the third and fourth terms on the right-hand side of (EC.15), and the second term on the right-hand side of (EC.20) vanish as n gets large. This argument formally justifies the approximation $X_k^n \approx X_k$ for $k \in \mathcal{S}$ as n gets large, where the X_k are independent Brownian motions with infinitesimal drift ν_k and infinitesimal variance σ_k^2 given as follows:

$$\nu_k := \begin{cases} -\eta_k \rho_k, & k \in \mathcal{S}^{\mathrm{MTO}}, \\ -\eta_k \rho_k + \sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \eta_j \lambda_j^\star / \mu_j, & k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \\ -\eta_k \lambda_k^\star / \mu_k, & k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}, \end{cases} \qquad \sigma_k^2 := \begin{cases} \lambda_k^\star \left(1 + c_{sk}^2\right), & k \in \mathcal{S}^{\mathrm{MTO}}, \\ \lambda_k^\star + (\lambda_k^\star + \sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \lambda_j^\star)\, c_{sk}^2, & k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \\ \lambda_k^\star, & k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}. \end{cases}$$

Using the formal limiting arguments above, we replace the processes Z_k^n, X_k^n, Y_k^n, I^n, and O_k^n in the nth system with their formal limits Z_k, X_k, Y_k, L, and O_k, respectively, which jointly satisfy the following:

$$\begin{aligned} Z_k(t) &= X_k(t) + \int_0^t \zeta_k(s)\, ds - \int_0^t \mathbf{1}_{\{k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}} \ell_k Z_k^+(s) ds - O_k(t) + \mu_k Y_k(t), \quad k \in \mathcal{S}^{\mathrm{MTO}}, \\ Z_k(t) &= X_k(t) + \int_0^t \zeta_k(s)\, ds - \int_0^t \ell_k Z_k^+(s)\, ds - O_k(t) + \mu_k Y_k(t) - \sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \mu_j Y_j(t), \quad k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \\ Z_k(t) &= X_k(t) + \int_0^t \zeta_k(s)\, ds - O_k(t) + \mu_k Y_k(t), \quad k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}, \\ L(t) &= \sum_{k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} Y_k(t). \end{aligned}$$

Similarly, (6)–(7) and (9)–(10) translate to the following conditions:

$$L \text{ and } O \text{ are nondecreasing with } L(0) = O(0) = 0,$$

$$Z_k(t) \geq 0 \text{ for all } k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}.$$

Finally, we consider the cumulative cost process in (EC.28). Observe that the second term on the right-hand side of (EC.28) contains the sojourn time term $w_k^n(t)$ for a product $k \in \mathcal{S}$ order accepted at time t. Applying the "snapshot principle" of Reiman (1984), we approximate $w_k^n(t)$ as follows:

$$\sqrt{n} w_k^n(t) \approx \frac{Q_k^n(t)}{\sqrt{n}\lambda_k^\star} = \frac{Z_k^n(t)}{\lambda_k^\star}, \quad t \geq 0,$$

which becomes more accurate as the system approaches heavy traffic. The snapshot principle states that in heavy traffic, the state of the system changes negligibly during the time an order spends in the system; see, e.g., Ata and Tongarlak (2013), Kim et al. (2018), and Liu and Sun (2022) for similar applications of the snapshot principle. Furthermore, it follows from the functional strong law of large numbers, (a loose application of) the random time change theorem, and the fact that the rejection process is expected to be of order $\sqrt{n}$ in the heavy-traffic regime that $\bar{A}_k^n(t) \approx \lambda_k^\star t$ as n gets large. Therefore, as n gets large,

$$\int_0^t \upsilon_k\left(\sqrt{n} w_k^n(s) - \delta_k\right) d\bar{A}_k^n(s) \approx \int_0^t \upsilon_k\left(\frac{Z_k^n(s)}{\lambda_k^\star} - \delta_k\right) d(\lambda_k^\star s) = \int_0^t \upsilon_k\left(Z_k^n(s) - \lambda_k^\star \delta_k\right) ds.$$

Using the above formal limiting arguments, we replace the cumulative cost process ξ^n in the nth system with its limiting process ξ, which is given as follows:

$$\xi(t) = -\frac{1}{2}\int_0^t \zeta(s)'\nabla^2\Pi(\lambda^\star)\zeta(s)\,ds + \sum_{k\in\mathcal{S}}\int_0^t \upsilon_k\big(Z_k(s)-\lambda_k^\star\delta_k\big)\,ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}}\int_0^t h_k Z_k^-(s)\,ds$$
$$+\sum_{k\in\mathcal{S}_{\mathrm{w}}}\int_0^t d_k\ell_k Z_k^+(s)\,ds + \sum_{k\in\mathcal{S}} r_k O_k(t).$$

EC.2. Solving the Bellman Equation

This section establishes the existence and uniqueness of the solution to the Bellman equation (55)–(56), effectively proving Theorem 1. We also provide an explicit characterization of the optimal value function and an optimal workload configuration function. To this end, we first analyze a class of Riccati equations in Appendix EC.2.1, which includes the Bellman equation, and provide an explicit solution for this class of equations. We then apply these results in Appendix EC.2.2 to construct a solution to the Bellman equation.

EC.2.1. Solution to a Special Class of Riccati Equations

In this section, we analyze an initial value problem (IVP) associated with a special class of Riccati equations. A Riccati equation is a first-order nonlinear ordinary differential equation of the form

$$y'(x) = f_2(x)y^2(x) + f_1(x)y(x) + f_0(x),$$

where f_0, f_1, and f_2 are given functions of x. The differential equation component of the Bellman equation (59)–(60) is a Riccati equation on specific intervals, and the results here facilitate the derivation of its solution; see Appendix EC.2.2 for details. To this end, we consider the following initial value problem:

$$y'(x) = c_4y^2(x) + (c_3x + c_2)y(x) + c_1x + c_0, \quad x \in [0,\infty), \tag{EC.29}$$

$$y(0) = y_0, \tag{EC.30}$$

where $y_0, c_0, c_1, c_2, c_3, c_4 \in \mathbb{R}$ are constants such that $c_4 \neq 0$.

LEMMA EC.1. *There exists a unique solution $y \in C^1[0,\infty)$ to (EC.29)–(EC.30).*

As a preliminary step in solving (EC.29)–(EC.30), we establish the following result, which provides an equivalence between this problem and a second-order initial value problem.

LEMMA EC.2. *For each $y \in C^1[0,\infty)$ satisfying (EC.29)–(EC.30), the function $z(x) := \exp\big(-c_4\int_0^x y(t)\,dt\big)$ for $x \in [0,\infty)$ satisfies*

$$z''(x) - (c_3x + c_2)z'(x) + c_4(c_1x + c_0)z(x) = 0, \quad x \in [0,\infty), \tag{EC.31}$$

$$z(0) = 1, \quad z'(0) = -c_4 y_0. \tag{EC.32}$$

Conversely, for each $z \in C^2[0,\infty)$ satisfying (EC.31)–(EC.32), the function $y(x) \coloneqq -z'(x)/(c_4 z(x))$ for $x \in [0,\infty)$ satisfies (EC.29)–(EC.30).

Next, we discuss the solution to (EC.29)–(EC.30). To simplify the analysis, we assume that $c_3 \geq 0$. This assumption entails no loss of generality, as the Bellman equation satisfies this condition. We proceed by considering two distinct cases: $c_3 = 0$ and $c_3 > 0$.

EC.2.1.1. Case 1: Solution for $\mathbf{c_3 = 0}$. When $c_3 = 0$, (EC.31) simplifies to the following:

$$z''(x) - c_2 z'(x) + c_4(c_1 x + c_0)z(x) = 0, \quad x \in [0,\infty). \tag{EC.33}$$

Following Ata and Barjesteh (2022, Appendix C), we write the solution to (EC.31) in terms of Airy functions. The Airy functions of the first kind $\mathrm{Ai} : \mathbb{R} \to \mathbb{R}$ and of the second kind $\mathrm{Bi} : \mathbb{R} \to \mathbb{R}$ are defined as

$$\mathrm{Ai}(x) \coloneqq \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}z^3 + zx\right) dz \quad \text{and} \quad \mathrm{Bi}(x) \coloneqq \frac{1}{\pi}\int_0^\infty \left(\cos\left(-\tfrac{1}{3}z^3 + zx\right) + \sin\left(\tfrac{1}{3}z^3 + zx\right)\right) dz, \quad x \in \mathbb{R}.$$

The next lemma provides a closed-form solution to (EC.29)–(EC.30) when $c_3 = 0$. To state it, define the linear function $u(x) \coloneqq \left[c_2^2 - 4c_4(c_1 x + c_0)\right] / 4(c_1 c_4)^{2/3}$ for $x \in \mathbb{R}$, and let C_1 and C_2 be constants given by

$$C_1 \coloneqq \frac{\mathrm{Bi}'(u(0)) - \mathrm{Bi}(u(0))(c_1 c_4)^{-1/3}(c_4 y_0 + \frac{c_2}{2})}{\mathrm{Ai}(u(0))\mathrm{Bi}'(u(0)) - \mathrm{Ai}'(u(0))\mathrm{Bi}(u(0))}, \quad C_2 \coloneqq \frac{-\mathrm{Ai}'(u(0)) + \mathrm{Ai}(u(0))(c_1 c_4)^{-1/3}(c_4 y_0 + \frac{c_2}{2})}{\mathrm{Ai}(u(0))\mathrm{Bi}'(u(0)) - \mathrm{Ai}'(u(0))\mathrm{Bi}(u(0))}.$$

LEMMA EC.3. *Suppose that $c_3 = 0$ and let $z \in C^2[0,\infty)$ be the function defined by*

$$z(x) \coloneqq C_1 \exp\left(\frac{c_2 x}{2}\right)\mathrm{Ai}(u(x)) + C_2 \exp\left(\frac{c_2 x}{2}\right)\mathrm{Bi}(u(x)), \quad x \in [0,\infty).$$

Then, the function $y(x) \coloneqq -z'(x)/(c_4 z(x))$ for $x \in [0,\infty)$ is the unique solution to (EC.29)–(EC.30).

Case 2: Solution for $\mathbf{c_3 > 0}$. When $c_3 > 0$, we proceed in a similar fashion as above, with the primary difference being that the second-order differential equation in Lemma EC.2 retains the constant c_3, resulting in a different solution. As shown in Zaitsev and Polyanin (2002, Equation 2.1.2.108) and Abramowitz and Stegun (1965, Chapter 13), the general solution to (EC.31) when $c_3 \neq 0$ is given by

$$z(x) = \exp(Cx)\, u\left(A, \frac{1}{2}, \frac{c_3}{2}(x - B)^2\right), \quad x \in [0,\infty), \tag{EC.34}$$

where $A \coloneqq -c_1^2 c_4^2 / 2c_3^3 + c_2 c_1 c_4 / 2c_3^2 - c_0 c_4 / 2c_3$, $B \coloneqq 2c_1 c_4 / c_3^2 + c_2 / c_3$, and $C \coloneqq c_1 c_4 / c_3$, and the function $x \mapsto u(a,b,x)$ for $x \in [0,\infty)$ is the general solution to the degenerate hypergeometric differential equation

$$xu''(x) + (b - x)u'(x) - au(x) = 0.$$

When b is not an integer, as is the case in (EC.34), the general solution to this hypergeometric equation can be expressed in terms of Kummer's function, defined as

$$\Phi(a,\, b,\, x) \coloneqq 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!}, \quad x \in [0,\infty),$$

where $(a)_0 \coloneqq 1$ and $(a)_k \coloneqq a(a+1)\cdots(a+k-1)$ for $k \in \mathbb{N}$. Kummer's function is an entire function of x (i.e., an analytic function on the entire real line), except when b is a negative integer; see, e.g., Kummer (1837) and Zaitsev and Polyanin (2002, Section S.2.7) for further details. Using Kummer's function, the general solution to the hypergeometric equation is given by

$$u(x) = C_1\Phi(a,\, b,\, x) + C_2 x^{1-b}\Phi(a-b+1,\, 2-b,\, x), \quad x \in [0,\infty), \tag{EC.35}$$

where C_1 and C_2 are arbitrary constants. The next lemma provides a closed-form solution to (EC.29)–(EC.30) when $c_3 \neq 0$. To state it, let C_1 and C_2 be the constants given by

$$C_1 = \begin{cases} \frac{G_0 + c_3 B^2 G_0' - B(C + c_4 y_0) G_0}{F_0(G_0 + c_3 B^2 G_0') - c_3 B^2 G_0 F_0'}, & B \neq 0, \\ 1, & B = 0, \end{cases} \quad \text{and} \quad C_2 = \begin{cases} \frac{C_1 F_0 - 1}{H_0 B G_0}, & B \neq 0, \\ -\frac{C + c_4 y_0}{H_0}, & B = 0 \end{cases} \tag{EC.36}$$

where $F_0 \coloneqq \Phi(A, 1/2, c_3B^2/2)$, $G_0 \coloneqq \Phi(A+1/2, 3/2, c_3B^2/2)$, $F_0' \coloneqq \Phi'(A, 1/2, c_3B^2/2)$, $G_0' \coloneqq \Phi'(A+1/2, 3/2, c_3B^2/2)$, and $H_0 = \sqrt{c_3/2}$.

LEMMA EC.4. *Suppose that $c_3 > 0$ and let $z \in C^2[0,\infty)$ be the function defined by*

$$z(x) \coloneqq \exp(Cx)\left(C_1\Phi\big(A, \frac{1}{2}, \frac{c_3}{2}(x-B)^2\big) + C_2\sqrt{\frac{c_3}{2}}\,(x-B)\Phi\big(A+\frac{1}{2}, \frac{3}{2}, \frac{c_3}{2}(x-B)^2\big)\right), \quad x \in [0,\infty).$$

Then, the function $y(x) \coloneqq -z'(x)/(c_4 z(x))$ for $x \in [0,\infty)$ is the unique solution to (EC.29)–(EC.30).

EC.2.2. Existence of a Solution to the Bellman Equation

In this section, we analyze two IVPs that are closely related to the Bellman equation (59)–(60) introduced in Section 6.2. As a preliminary step, we first establish key properties of the effective state cost function in Appendix EC.2.2.1 and then present additional auxiliary results in Appendix EC.2.2.2 to facilitate the analysis of the IVPs. Finally, in Appendix EC.2.2.3, we introduce the IVPs and leverage the results from Appendices EC.2.2.1 and EC.2.2.2 to prove the existence of a unique solution to the Bellman equation.

EC.2.2.1. Characterization of the Effective State Cost Function. To facilitate the analysis to follow, we recall key definitions from Section 6.2. First, the effective state cost function $\phi : \mathbb{R}^2 \to \mathbb{R}$, originally defined in (58), is restated here for convenience:

$$\phi(w, y) \coloneqq \min_{z \in \mathcal{A}(w)} \varphi(z, y), \quad w, y \in \mathbb{R},$$

where $\mathcal{A}(w)$ denotes the set of admissible workload distribution vectors at workload level w, as defined in (42). Second, the function $\varphi : \mathbb{R}^K \times \mathbb{R} \to \mathbb{R}$, originally defined in (54), is restated here for convenience:

$$\varphi(z,y) \coloneqq \sum_{k\in\mathcal{S}_{\mathrm{w}}} \ell_k\big(d_k - m_k y\big) z_k^+ + \sum_{k\in\mathcal{S}} \upsilon_k\big(z_k - \lambda_k^\star \delta_k\big) + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} h_k z_k^-, \quad (z,y)\in\mathbb{R}^K\times\mathbb{R},$$

where the cost function υ_k for $k\in\mathcal{S}$ is given by (12). Third, the cost terms $\hat{\alpha}_k$ and $\hat{\beta}_k$, originally defined in (65), are restated here for convenience:

$$\hat{\alpha}_k(y) \coloneqq \begin{cases} \alpha_k + \ell_k(d_k - m_k y), & k\in\mathcal{S}_{\mathrm{w}}, \\ \alpha_k, & k\in\mathcal{S}_{\mathrm{o}}, \end{cases} \quad \text{and} \quad \hat{\beta}_k \coloneqq \begin{cases} \beta_k, & k\in\mathcal{S}_{\mathrm{o}}\cup\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}, \\ h_k, & k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}. \end{cases}$$

The corresponding cost functions $\hat{\upsilon}_k : \mathbb{R}^2 \to \mathbb{R}$ for $k\in\mathcal{S}$ (comprising tardiness, earliness, holding, and abandonment costs), originally defined in (66), are restated here for convenience:

$$\hat{\upsilon}_k(x,y) \coloneqq \begin{cases} \hat{\alpha}_k(y)\,x, & x\ge 0, \\ -\hat{\beta}_k\,x, & x<0. \end{cases}$$

Finally, the function φ, which was rewritten in (67), is restated here for convenience:

$$\varphi(z,y) = \sum_{k\in\mathcal{S}_{\mathrm{o}}} \hat{\upsilon}_k(z_k - \lambda_k^\star\delta_k, y) + \sum_{k\in\mathcal{S}_{\mathrm{w}}} \hat{\upsilon}_k(z_k, y), \quad (z,y)\in\mathbb{R}^K\times\mathbb{R}.$$

The next two results establish that as long as $y\le\kappa$, the workload cannot be distributed in a way that results in a negative effective state cost.[1] The restriction $y\le\kappa$ also allows us to derive a closed-form expression for ϕ, which facilitates the analysis in Appendix EC.2.2.3. Nevertheless, as noted in the footnote, this restriction is harmless for our purposes.

LEMMA EC.5. *For all $k\in\mathcal{S}$ and $y\in(-\infty,\kappa]$, we have that $\hat{\alpha}_k(y)>0$ and $\hat{\beta}_k\ge 0$, with $\hat{\beta}_k>0$ for all $k\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}\cup\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$.*

COROLLARY EC.1. *For all $w\in\mathbb{R}$ and $y\in(-\infty,\kappa]$, we have that $\phi(w,y)\ge 0$.*

We next provide a closed-form characterization of the effective state cost function ϕ when $y\in(-\infty,\kappa]$ by means of identifying a minimizer on the right-hand side of (58). Since here we focus solely on the mathematical results, we encourage the reader to refer to Section 6.3—in particular, Figures 2 and 3—for a more detailed intuitive explanation of the mathematical expressions that follow. To that end, recall the constants w_0 and w_1, originally defined in (61), which are restated here for convenience:

$$w_0 \coloneqq \sum_{k\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}} m_k\lambda_k^\star\delta_k \quad \text{and} \quad w_1 \coloneqq \sum_{k\in\mathcal{S}_{\mathrm{o}}} m_k\lambda_k^\star\delta_k.$$

[1] The restriction $y\le\kappa$ is needed because if $y>\kappa$, there could exist a class $k\in\mathcal{S}_{\mathrm{w}}$ such that $d_k - m_k y < 0$. In this case, one could increase z_k^+ without bound, leading to an unbounded state cost from below. This restriction is innocuous, as the optimal value function (as we will show later) satisfies $v(w)\le\kappa$ for $w\in[l,u]$.

These constants are introduced because the structure of the effective state cost function and its minimizer differs across the three intervals $(-\infty, w_0)$, $[w_0, w_1]$, and (w_1, ∞). Before stating the results, we define the (not necessarily unique) bijective mapping $i^\star : \{1, \ldots, |\mathcal{S}_{\text{o}}^{\text{MTO}}| + |\mathcal{S}_{\text{w}}^{\text{MTS}}|\} \to \mathcal{S}_{\text{o}}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$ that orders the online MTO and walk-in MTS products in increasing order of $\hat{\beta}_k / m_k$, so that

$$\hat{\beta}_{i^\star(k)} / m_{i^\star(k)} \le \hat{\beta}_{i^\star(k+1)} / m_{i^\star(k+1)} \quad \text{for} \quad k = 1, \ldots, |\mathcal{S}_{\text{o}}^{\text{MTO}}| + |\mathcal{S}_{\text{w}}^{\text{MTS}}| - 1.$$

We then define $I \coloneqq \min\{1 \le k \le |\mathcal{S}_{\text{o}}^{\text{MTO}}| + |\mathcal{S}_{\text{w}}^{\text{MTS}}| : i^\star(k) \in \mathcal{S}_{\text{w}}^{\text{MTS}}\}$, which corresponds to the index of the cheapest walk-in MTS product according to the ordering above. In particular, $i^\star(I) \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ is the cheapest walk-in MTS product according to the ordering above, while products $i^\star(j) \in \mathcal{S}_{\text{o}}^{\text{MTO}}$ for $j = 1, \ldots, I-1$ are the online MTO products that are cheaper than $i^\star(I)$ in the same sense. We now define the product class $k^\star(w, y) \in \mathcal{S}$ for $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$ as follows:

$$k^\star(w,y) \coloneqq \begin{cases} \max\left\{1 \le k \le I : \sum_{j=1}^{k-1} m_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)} \le w_0 - w\right\}, & w \in (-\infty, w_0), \\ \min\left\{k \in \mathcal{S}_{\text{o}}^{\text{MTS}} : \sum_{j \in \mathcal{S}_{\text{o}}^{\text{MTS}} : j \le k} m_j \lambda^\star_j \delta_j > w - w_0\right\}, & w \in [w_0, w_1], \\ \arg\min_{k \in \mathcal{S}} \left\{\hat{\alpha}_k(y) / m_k\right\}, & w \in (w_1, \infty). \end{cases} \tag{EC.37}$$

Notice that $k^\star(w, y)$ depends only on w when $w \in (-\infty, w_1]$ and depends only on y when $w \in (w_1, \infty)$.[2] It is straightforward to verify that we can equivalently, and more conveniently, express $k^\star(w, y)$ for $w \in (-\infty, w_0)$ and $y \in (-\infty, \kappa]$ as follows:

$$k^\star(w,y) = \begin{cases} I - k, & w \in [\,\tau^-_{I-k}\,, \tau^-_{I-k-1}), \quad k = 1, \ldots, I-1, \\ I, & w \in (-\infty\,, \tau^-_{I-1}), \end{cases} \tag{EC.38}$$

where the interval endpoints τ^-_k are given as follows:

$$\tau^-_k \coloneqq \begin{cases} w_0, & k = 0, \\ w_0 - \sum_{j=1}^{k} m_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)}, & k = 1, \ldots, I-1. \end{cases} \tag{EC.39}$$

A more detailed description of $k^\star(w, y)$ for $w \in (w_1, \infty)$ and $y \in (-\infty, \kappa]$ is provided in Appendix EC.2.2.2.

With the above notation established, we next identify a minimizer of the right-hand side of (58) when $y \in (-\infty, \kappa]$. Define the workload configuration $\mathcal{Z} : \mathbb{R} \times (-\infty, \kappa] \to \mathbb{R}^K$ as follows: For $w \in (-\infty, w_0)$, the workload configuration is given by

$$\mathcal{Z}_k(w,y) \coloneqq \begin{cases} \lambda^\star_k \delta_k, & k \in \mathcal{S}_{\text{o}}^{\text{MTO}} \setminus \{i^\star(1), \ldots, i^\star(k^\star(w,y))\}, \\ \lambda^\star_k \delta_k - (\tau^-_{k^\star(w,y)-1} - w)/m_k, & k = i^\star(k^\star(w,y)), \\ 0, & \text{otherwise}. \end{cases} \tag{EC.40}$$

[2] Equation (EC.37) provides a single, unified definition of the product class $k^\star(w, y)$ for all workload levels, rather than defining it separately for different workload intervals. However, when the context permits, we may drop y (resp., w) and write $k^\star(w)$ (resp., $k^\star(y)$) in favor of $k^\star(w, y)$ when $w \le w_1$ (resp., $w > w_1$).

For $w \in [w_0, w_1]$, the workload configuration is given by

$$\mathcal{Z}_k(w,y) := \begin{cases} \lambda_k^\star \delta_k, & k \in \mathcal{S}_\mathrm{o}^{\mathrm{MTO}} \cup \{j \in \mathcal{S}_\mathrm{o}^{\mathrm{MTS}} : j < k^\star(w,y)\}, \\ (w - w_0 - \sum_{j \in \mathcal{S}_\mathrm{o}^{\mathrm{MTS}} : j < k^\star(w,y)} m_j \lambda_j^\star \delta_j)/m_k, & k = k^\star(w,y), \\ 0, & \text{otherwise.} \end{cases} \tag{EC.41}$$

For $w \in (w_1, \infty)$, the workload configuration is given by

$$\mathcal{Z}_k(w,y) := \begin{cases} \lambda_k^\star \delta_k + (w - w_1)/m_k, & k = k^\star(w,y), \\ \lambda_k^\star \delta_k, & \text{otherwise.} \end{cases} \tag{EC.42}$$

The following lemma shows that the workload configuration defined above is an optimal workload configuration, as it minimizes the effective state costs.

LEMMA EC.6. *The workload configuration $\mathcal{Z}$ defined in (EC.40)–(EC.42) satisfies $\mathcal{Z}(w,y) \in \arg\min_{z \in \mathcal{A}(w)} \varphi(z,y)$ for all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$, i.e., is an optimal workload configuration. Moreover, the optimal workload configuration need not be unique in general.*

The following corollary provides a closed-form expression for the effective state cost function.

COROLLARY EC.2. *For $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$, the effective state cost function $\phi(w,y)$ is given by*

$$\phi(w,y) = \begin{cases} \hat{\beta}_{i^\star(k^\star(w,y))}(\tau^-_{k^\star(w,y)-1} - w)/m_{i^\star(k^\star(w,y))} + \sum_{j=1}^{k^\star(w,y)-1} \hat{\beta}_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)}, & w \in (-\infty, w_0), \\ 0, & w \in [w_0, w_1], \\ \hat{\alpha}_{k^\star(w,y)}(y)(w - w_1)/m_{k^\star(w,y)}, & w \in (w_1, \infty). \end{cases}$$

The following lemma establishes some structural properties of the effective state cost function, which will be used in Appendix EC.2.2.3.

LEMMA EC.7. *For each $y \in (-\infty, \kappa]$, the mapping $w \mapsto \phi(w,y)$ has the following properties:*

(a) It is continuous everywhere and differentiable almost everywhere with respect to Lebesgue measure.

(b) It is strictly decreasing on $(-\infty, w_0)$ with $\phi(w,y) \to \infty$ as $w \to -\infty$, constant on $[w_0, w_1]$ with $\phi(w,y) \equiv 0$, and strictly increasing on (w_1, ∞) with $\phi(w,y) \to \infty$ as $w \to \infty$.

EC.2.2.2. Characterization of the Product Class $k^\star(w,y)$ for $w \in (w_1, \infty)$. In this section, we further analyze the product class $k^\star(w,y)$ defined in (EC.37), focusing specifically on the case where $w \in (w_1, \infty)$ and $y \in (-\infty, \kappa]$. An explicit characterization of this class will be useful for solving the Bellman equation later. To facilitate the analysis, define the function $\mathbf{A} : (-\infty, \kappa] \to \mathbb{R}$ as follows:

$$\mathbf{A}(y) := \min_{k \in \mathcal{S}} \{\hat{\alpha}_k(y)/m_k\}, \quad y \in (-\infty, \kappa]. \tag{EC.43}$$

Moreover, we introduce the set $\mathcal{S}^0 := \mathcal{S}_\mathrm{o} \cup \{k \in \mathcal{S}_\mathrm{w} : \ell_k = 0\}$, which consists of all products with a zero abandonment rate. The following result provides some useful structural properties of $\mathbf{A}$.

LEMMA EC.8. *The function* $\mathbf{A} : (-\infty, \kappa] \to \mathbb{R}$ *is strictly positive, nonincreasing, continuous, concave, and non-differentiable at at most finitely many points. Furthermore, it is either constant over its entire domain or remains constant up to its first point of non-differentiability, after which it has a strictly negative derivative at all points where it is differentiable.*

By Lemma EC.8, the number of points where $\mathbf{A}$ is non-differentiable is finite; denote this number by $b \in \mathbb{N}$. Setting $y_0 := \kappa$, we define the products k_j and points y_j iteratively as follows:[3]

$$k_j := \operatorname*{arg\,min}_{k \in \mathcal{S} \setminus \{k_0, \ldots, k_{j-1}\}} \{\hat{\alpha}_k(y_j)/m_k\}, \quad j = 0, 1, \ldots, b, \tag{EC.44}$$

$$y_j := \sup\{y < y_{j-1} : \hat{\alpha}_k(y)/m_k < \hat{\alpha}_{k_j}(y)/m_{k_j} \text{ for some } k \in \mathcal{S}\}, \quad j = 1, \ldots, b. \tag{EC.45}$$

It follows that $k_0, \ldots, k_{b-1} \in \{k \in \mathcal{S}_{\mathrm{w}} : \ell_k > 0\}$ and $k_b \in \mathcal{S}^0$. (Note that the points y_j given by (EC.45) are precisely those where the derivative of $\mathbf{A}$ changes.) We now provide a more explicit characterization of the product class $k^\star(w, y)$ for $w \in (w_1, \infty)$ and $y \in (-\infty, \kappa]$ and the function $\mathbf{A}$.

COROLLARY EC.3. *The product class* $k^\star(w, y)$ *for* $w \in (w_1, \infty)$ *and* $y \in (-\infty, \kappa]$, *and the function* $\mathbf{A} : (-\infty, \kappa] \to \mathbb{R}$, *can be equivalently expressed as follows:*

$$k^\star(w, y) = \begin{cases} k_j, & y \in (y_{j+1}, y_j], \quad j = 0, 1, \ldots, b-1, \\ k_b, & y \in (-\infty, y_b], \end{cases} \tag{EC.46}$$

$$\mathbf{A}(y) = \begin{cases} \hat{\alpha}_{k_j}(y)/m_{k_j}, & y \in (y_{j+1}, y_j], \quad j = 0, 1, \ldots, b-1, \\ \alpha_{k_b}/m_{k_b}, & y \in (-\infty, y_b]. \end{cases}$$

EC.2.2.3. Two Initial Value Problems Related to the Bellman Equation. This section analyzes two IVPs that are closely related to the Bellman equation (59)–(60). The first IVP constructs a solution on $(-\infty, w_1]$, while the second IVP does so on $[w_1, \infty)$. By pasting together these solutions, we prove the existence of a solution to the Bellman equation, thereby proving Theorem 1. Throughout, we attach the superscripts "−" and "+" to various quantities associated with the first and second IVPs, respectively.

EC.2.2.3.1. IVP on $(-\infty, w_1]$. For each $\gamma \geq 0$, consider the following IVP:

$$v'(w) = \frac{m' H^{-1} m}{2\sigma^2} v^2(w) - \frac{2\mu}{\sigma^2} v(w) - \frac{2}{\sigma^2} \phi(w, v(w)) + \frac{2\gamma}{\sigma^2}, \quad w \in [l_\gamma, w_1], \tag{EC.47}$$

$$v(l_\gamma) = 0, \tag{EC.48}$$

[3] Whenever (EC.44) does not have a unique minimizer, choose the product k_j for which $\hat{\alpha}_{k_j}(y)/m_{k_j} < \hat{\alpha}_k(y)/m_k$ for all $(k, y) \in \mathcal{S} \setminus \{k_0, \ldots, k_{j-1}, k_j\} \times (y_j - \epsilon, y_j)$ for some sufficiently small $\epsilon > 0$. (If such a product k_j does not exist, there exist $n_j \geq 2$ products $k_j(1), \ldots, k_j(n_j)$ such that $\hat{\alpha}_{k_j(1)}(y)/m_{k_j(1)} = \cdots = \hat{\alpha}_{k_j(n_j)}(y)/m_{k_j(n_j)}$ for all $y \in \mathbb{R}$ and $\hat{\alpha}_{k_j(1)}(y)/m_{k_j(1)} < \hat{\alpha}_k(y)/m_k$ for all $(k, y) \in \mathcal{S} \setminus \{k_0, \ldots, k_{j-1}, k_j(1), \ldots, k_j(n_j)\} \times (y_j - \epsilon, y_j)$ for some sufficiently small $\epsilon > 0$. In this case, we arbitrarily choose one of the $k_j(i)$ as the minimizer in (EC.44).) This ensures that $y_j \neq y_{j-1}$, guaranteeing that all breakpoints of $\mathbf{A}$ are captured in (EC.45).

where the lower barrier $l_\gamma \in \mathbb{R}$ is given by[4]

$$l_\gamma := \sup\left\{l \le w_0 : \phi(l,0) = \gamma\right\}. \tag{EC.49}$$

It follows from (EC.49) that any solution $v \in C^1[l_\gamma, w_1]$ to (EC.47)–(EC.48) satisfies $v'(l_\gamma) = 0$. Therefore, we interpret the solution to this IVP as the value function associated with a barrier policy (with a lower barrier at l_γ) that achieves a long-run average cost of γ and satisfies the smooth pasting condition at the lower barrier; see Proposition 2.

We next prove the existence of a closed-form solution to the IVP (EC.47)–(EC.48) when γ is not too large.[5] In particular, the solution is constructed iteratively over subintervals where the effective state cost function ϕ is linear, allowing us to leverage the results from Appendix EC.2.1 to derive a closed-form solution. To that end, we partition the interval $[l_\gamma, w_1]$ as follows:

$$[\,l_\gamma\,,\,w_1] = [\,l_\gamma\,,\,\tau^-_{k^\star(l_\gamma)-1}) \cup \bigcup_{k=1}^{k^\star(l_\gamma)-1} [\tau^-_{k^\star(l_\gamma)-k}\,,\,\tau^-_{k^\star(l_\gamma)-k-1}) \cup [w_0\,,\,w_1], \tag{EC.50}$$

where $\tau^-_{k^\star(l_\gamma)} := l_\gamma$ for notational convenience. (Note that since $k^\star(w,y)$ only depends on w (and not y) for $w \in (-\infty, w_1]$, we write $k^\star(l_\gamma)$ for simplicity; see (EC.37).) We proceed in four main steps. In Step 1, we find the unique solution on $[l_\gamma, \tau^-_{k^\star(l_\gamma)-1}]$. In Step 2, we use the value of the solution from Step 1 at $\tau^-_{k^\star(l_\gamma)-1}$ as the initial condition and find the unique solution on $[\tau^-_{k^\star(l_\gamma)-1}, \tau^-_{k^\star(l_\gamma)-2}]$. This process continues iteratively, where the solution from the previous interval provides the initial condition for the next, until we obtain unique solutions on $[\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1}]$ for $k = 1, \ldots, k^\star(l_\gamma) - 1$. In Step 3, we use the value of the last solution from Step 2 at w_0 as the initial condition and find the unique solution on $[w_0, w_1]$. In Step 4, we combine the solutions from Steps 1–3 to obtain a continuously differentiable function on $[l_\gamma, w_1]$. After constructing this function, we establish its key properties and show that for γ belonging to a certain compact set, it uniquely solves the IVP (EC.47)–(EC.48).

Step 1: Solution on $\left[l_\gamma, \tau^-_{k^\star(l_\gamma)-1}\right)$. We construct a solution on the interval $[l_\gamma, \tau^-_{k^\star(l_\gamma)-1})$ by considering the following IVP:

$$v'(w) = \frac{m'H^{-1}m}{2\sigma^2}\,v^2(w) - \frac{2\mu}{\sigma^2}\,v(w) - \frac{2}{\sigma^2}\,\phi^-_{k^\star(l_\gamma)}(w) + \frac{2\gamma}{\sigma^2}, \quad w \in [l_\gamma, \infty), \tag{EC.51}$$

$$v(l_\gamma) = 0, \tag{EC.52}$$

[4] The existence of l_γ follows from Lemma EC.7, since the mapping $w \mapsto \phi(w,0)$ is continuous and strictly decreasing on $(-\infty, w_0]$, with $\phi(w_0, 0) = 0$ and $\phi(w,0) \to \infty$ as $w \to -\infty$.

[5] This statement will be made precise later. The heart of the matter is that as long as γ remains within an appropriate range, the solution to the IVP (EC.47)–(EC.48) will be (pointwise) bounded above by κ, allowing us to apply the closed-form expression for $\phi(w,y)$ when $y \in (-\infty, \kappa]$.

where

$$\phi^-_{k^\star(l_\gamma)}(w) \coloneqq \frac{\hat{\beta}_{i^\star(k^\star(l_\gamma))}}{m_{i^\star(k^\star(l_\gamma))}} \left(\tau^-_{k^\star(l_\gamma)-1} - w\right) + \sum_{j=1}^{k^\star(l_\gamma)-1} \hat{\beta}_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)}, \quad w \in [l_\gamma, \infty), \tag{EC.53}$$

is the linear function that coincides with $\phi(w, y)$ for $w \in [l_\gamma, \tau^-_{k^\star(l_\gamma)-1}]$ and $y \in (-\infty, \kappa]$. Since $\phi^-_{k^\star(l_\gamma)}$ is a linear function, we apply the results from Appendix EC.2.1 to obtain a solution to (EC.51)–(EC.52). To be specific, consider the Riccati equation (EC.29)–(EC.30) with the following constants:

$$c_0 = -\frac{2}{\sigma^2}\left[\frac{\hat{\beta}_{i^\star(k^\star(l_\gamma))}}{m_{i^\star(k^\star(l_\gamma))}} \left(\tau^-_{k^\star(l_\gamma)-1} + l_\gamma\right) + \sum_{j=1}^{k^\star(l_\gamma)-1} \hat{\beta}_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)} - \gamma\right],$$

$$c_1 = \frac{2}{\sigma^2}\frac{\hat{\beta}_{i^\star(k^\star(l_\gamma))}}{m_{i^\star(k^\star(l_\gamma))}}, \quad c_2 = -\frac{2\mu}{\sigma^2}, \quad c_3 = 0, \quad c_4 = \frac{m' H^{-1} m}{2\sigma^2}, \quad \text{and} \quad y_0 = 0.$$

By Lemma EC.3, there exists a unique solution $y^-_{\gamma, k^\star(l_\gamma)} \in C^1[0, \infty)$ to this Riccati equation. It then follows that the function $\hat{v}^-_{\gamma, k^\star(l_\gamma)} \in C^1[l_\gamma, \infty)$ given by

$$\hat{v}^-_{\gamma, k^\star(l_\gamma)}(w) \coloneqq y^-_{\gamma, k^\star(l_\gamma)}(w - l_\gamma), \quad w \in [l_\gamma, \infty), \tag{EC.54}$$

is the unique solution to (EC.51)–(EC.52).

Step 2: Solution on $[\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1})$ for $k = 1, \ldots, k^\star(l_\gamma) - 1$. Given the solution $\hat{v}^-_{\gamma, k^\star(l_\gamma)-k+1}$ on the interval $[\tau^-_{k^\star(l_\gamma)-k+1}, \tau^-_{k^\star(l_\gamma)-k})$ for $k = 1, \ldots, k^\star(l_\gamma) - 1$, we construct a solution on the interval $[\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1})$ by considering the following IVP:

$$v'(w) = \frac{m' H^{-1} m}{2\sigma^2} v^2(w) - \frac{2\mu}{\sigma^2} v(w) - \frac{2}{\sigma^2} \phi^-_{k^\star(l_\gamma)-k}(w) + \frac{2\gamma}{\sigma^2}, \quad w \in [\tau^-_{k^\star(l_\gamma)-k}, \infty), \tag{EC.55}$$

$$v(\tau^-_{k^\star(l_\gamma)-k}) = \hat{v}^-_{\gamma, k^\star(l_\gamma)-k+1}(\tau^-_{k^\star(l_\gamma)-k}), \tag{EC.56}$$

where

$$\phi^-_{k^\star(l_\gamma)-k}(w) \coloneqq \frac{\hat{\beta}_{i^\star(k^\star(l_\gamma)-k)}}{m_{i^\star(k^\star(l_\gamma)-k)}} \left(\tau^-_{k^\star(l_\gamma)-k-1} - w\right) + \sum_{j=1}^{k^\star(l_\gamma)-k-1} \hat{\beta}_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)}, \quad w \in [\tau^-_{k^\star(l_\gamma)-k}, \infty), \tag{EC.57}$$

is the linear function that coincides with $\phi(w, y)$ for $w \in [\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1}]$ and $y \in (-\infty, \kappa]$. As in Step 1, we apply the results from Appendix EC.2.1 to obtain a solution. To be specific, consider the Riccati equation (EC.29)–(EC.30) with the following constants:

$$c_0 = -\frac{2}{\sigma^2}\left(\frac{\hat{\beta}_{i^\star(k^\star(l_\gamma)-k)}}{m_{i^\star(k^\star(l_\gamma)-k)}} \left(\tau^-_{k^\star(l_\gamma)-k-1} + \hat{v}^-_{\gamma, k^\star(l_\gamma)-k+1}(\tau^-_{k^\star(l_\gamma)-k-1})\right) + \sum_{j=1}^{k^\star(l_\gamma)-k-1} \hat{\beta}_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)} - \gamma\right),$$

$$c_1 = \frac{2}{\sigma^2}\frac{\hat{\beta}_{i^\star(k^\star(l_\gamma)-k)}}{m_{i^\star(k^\star(l_\gamma)-k)}}, \quad c_2 = -\frac{2\mu}{\sigma^2}, \quad c_3 = 0, \quad c_4 = \frac{m' H^{-1} m}{2\sigma^2}, \quad \text{and} \quad y_0 = \hat{v}^-_{\gamma, k^\star(l_\gamma)-k+1}(\tau^-_{k^\star(l_\gamma)-k}).$$

By Lemma EC.3, there exists a unique solution $y^-_{\gamma,k^\star(l_\gamma)-k} \in C^1[0,\infty)$ to this Riccati equation. It then follows that the function $\hat{v}^-_{\gamma,k^\star(l_\gamma)-k} \in C^1[\tau^-_{k^\star(l_\gamma)-k},\infty)$ given by

$$\hat{v}^-_{\gamma,k^\star(l_\gamma)-k}(w) \coloneqq y^-_{\gamma,k^\star(l_\gamma)-k}(w-\tau^-_{k^\star(l_\gamma)-k}), \quad w \in [\tau^-_{k^\star(l_\gamma)-k},\infty), \tag{EC.58}$$

is the unique solution to (EC.55)–(EC.56).

Step 3: Solution on $[w_0, w_1]$**.** Given the solution $\hat{v}^-_{\gamma,1}$ on the interval $[\tau_1, w_0)$, we construct a solution on the interval $[w_0, w_1]$ by considering the following IVP:

$$v'(w) = \frac{m'H^{-1}m}{2\sigma^2}v^2(w) - \frac{2\mu}{\sigma^2}v(w) + \frac{2\gamma}{\sigma^2}, \quad w \in [w_0,\infty), \tag{EC.59}$$

$$v(w_0) = \hat{v}^-_{\gamma,1}(w_0). \tag{EC.60}$$

Similar to Steps 1–2, we apply the results from Appendix EC.2.1 to obtain a solution. To be specific, consider the Riccati equation (EC.29)–(EC.30) with the following constants:

$$c_0 = \frac{2\gamma}{\sigma^2}, \quad c_1 = 0, \quad c_2 = -\frac{2\mu}{\sigma^2}, \quad c_3 = 0, \quad c_4 = \frac{m'H^{-1}m}{2\sigma^2}, \quad \text{and} \quad y_0 = \hat{v}^-_{\gamma,1}(w_0).$$

By Lemma EC.3, there exists a unique solution $y^-_{\gamma,0} \in C^1[0,\infty)$ to this Riccati equation. It then follows that the function $\hat{v}^-_{\gamma,0} \in C^1[w_0,\infty)$ given by

$$\hat{v}^-_{\gamma,0}(w) \coloneqq y^-_{\gamma,0}(w-w_0), \quad w \in [w_0,\infty), \tag{EC.61}$$

is the unique solution to (EC.59)–(EC.60).

Step 4: Solution on $[l_\gamma, w_1]$**.** For each $\gamma \geq 0$, we define the function $v^-_\gamma : [l_\gamma, w_1] \to \mathbb{R}$ by pasting together the solutions from Steps 1–3 as follows:

$$v^-_\gamma(w) \coloneqq \begin{cases} \hat{v}^-_{\gamma,k^\star(l_\gamma)}(w), & w \in [l_\gamma, \tau^-_{k^\star(l_\gamma)-1}), \\ \hat{v}^-_{\gamma,k^\star(l_\gamma)-k}(w), & w \in [\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1}), \quad k = 1,\ldots,k^\star(l_\gamma)-1, \\ \hat{v}^-_{\gamma,0}(w), & w \in [w_0, w_1], \end{cases} \tag{EC.62}$$

where the functions $\hat{v}^-_{\gamma,k^\star(l_\gamma)-k}$ are given by (EC.54), (EC.58), and (EC.61). Next, we show that v^-_γ solves (EC.47)–(EC.48) for appropriate values of γ. As a preliminary, note that by Corollary EC.2, the effective state cost function $\phi(w,y)$ for $w \in [l_\gamma, w_1]$ and $y \in (-\infty,\kappa]$ can be rewritten as follows:

$$\phi(w,y) = \begin{cases} \phi^-_{k^\star(l_\gamma)}(w), & w \in [l_\gamma, \tau^-_{k^\star(l_\gamma)-1}), \\ \phi^-_{k^\star(l_\gamma)-k}(w), & w \in [\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1}), \quad k = 1,\ldots,k^\star(l_\gamma)-1, \\ 0, & w \in [w_0, w_1], \end{cases} \tag{EC.63}$$

where the functions ϕ^-_k are given by (EC.53) and (EC.57). We now consider the following IVP:

$$v'(w) = \frac{m'H^{-1}m}{2\sigma^2}v^2(w) - \frac{2\mu}{\sigma^2}v(w) - \frac{2}{\sigma^2}\phi(w,0) + \frac{2\gamma}{\sigma^2}, \quad w \in [l_\gamma, w_1], \tag{EC.64}$$

$$v(l_\gamma) = 0. \tag{EC.65}$$

LEMMA EC.9. *For each $\gamma \geq 0$, the function $v_\gamma^- : [l_\gamma, w_1] \to \mathbb{R}$ defined by (EC.62) is continuously differentiable and is the unique solution to (EC.64)–(EC.65).*

Next, we establish several structural properties of v_γ^- that will be essential in proving that it satisfies the IVP (EC.47)–(EC.48), as well as the existence and uniqueness of a solution to the Bellman equation (59)–(60).

LEMMA EC.10. *For each $\gamma > 0$, the function $v_\gamma^- \in C^1[l_\gamma, w_1]$ satisfies $(v_\gamma^-)'(l_\gamma) = 0$ and $(v_\gamma^-)'(w) > 0$ for all $w \in (l_\gamma, w_1]$. Consequently, v_γ^- is strictly increasing for each $\gamma > 0$.*

LEMMA EC.11. *The mapping $\gamma \mapsto v_\gamma^-$ is strictly increasing in the following sense: For $0 \leq \gamma_1 < \gamma_2$, we have that $v_{\gamma_2}^-(w) > v_{\gamma_1}^-(w)$ for all $w \in [l_{\gamma_1}, w_1]$.*

LEMMA EC.12. *The mapping $\gamma \mapsto v_\gamma^-(w_1)$ is strictly increasing with $v_0^-(w_1) = 0$ and $\lim_{\gamma\to\infty} v_\gamma^-(w_1) = \infty$.*

LEMMA EC.13. *The mapping $\gamma \mapsto v_\gamma^-(w_1)$ is continuous on $[0, \infty)$.*

Equipped with the above structural properties of v_γ^-, we now show that there exists a solution to the IVP (EC.47)–(EC.48). To that end, Lemma EC.12 ensures that the following is well-defined:

$$\gamma_\kappa := \sup\{\gamma \geq 0 : v_\gamma^-(w_1) \leq \kappa\}. \tag{EC.66}$$

The next result shows that γ_κ serves as an upper bound on the values of γ for which we can prove the existence of a solution to the IVP (EC.47)–(EC.48). However, as we will see in the subsequent sections, this restriction on γ does not prohibit us from finding the unique solution to the Bellman equation (59)–(60).

LEMMA EC.14. *For $\gamma \in [0, \gamma_\kappa]$, the function $v_\gamma^- \in C^1[l_\gamma, w_1]$ is the unique solution to (EC.47)–(EC.48).*

EC.2.2.3.2. IVP on $[w_1, \infty)$. For each $\gamma \geq 0$, consider the following IVP:

$$v'(w) = \frac{m'H^{-1}m}{2\sigma^2} v^2(w) - \frac{2\mu}{\sigma^2} v(w) - \frac{2}{\sigma^2} \phi(w, v(w)) + \frac{2\gamma}{\sigma^2}, \quad w \in [w_1, u_\gamma], \tag{EC.67}$$

$$v(u_\gamma) = \kappa, \tag{EC.68}$$

where the upper barrier $u_\gamma \in \mathbb{R}$ is given by[6]

$$u_\gamma := \inf\left\{w \geq w_1 : \phi(w, \kappa) = \gamma + \frac{m'H^{-1}m}{4}\kappa^2 - \mu\kappa\right\}. \tag{EC.69}$$

[6] Since $\gamma + \frac{m'H^{-1}m}{4}\kappa^2 - \mu\kappa > 0$ for all $\gamma \geq 0$, the existence of u_γ follows from Lemma EC.7 and the fact that the mapping $w \mapsto \phi(w, \kappa)$ is continuous and strictly increasing on $[w_1, \infty)$, with $\phi(w_1, \kappa) = 0$ and $\phi(w, \kappa) \to \infty$ as $w \to \infty$. This also ensures that $u_\gamma > w_1$ for all $\gamma \geq 0$.

It follows from (EC.69) that any solution $v \in C^1[w_1, u_\gamma]$ to (EC.67)–(EC.68) satisfies $v'(u_\gamma) = 0$. Therefore, we interpret the solution to this IVP as the value function associated with a barrier policy (with an upper barrier at u_γ) that achieves a long-run average cost of γ and satisfies the smooth pasting condition at the upper barrier; see Proposition 2.

We next prove the existence of a closed-form solution to the IVP (EC.67)–(EC.68) for all $\gamma \geq 0$. In particular, the solution is constructed iteratively over subintervals where the product class $k^\star(w, \cdot)$ for $w \in [w_1, \infty)$ remains constant, allowing us to leverage the results from Appendix EC.2.1 to derive a closed-form solution. We proceed in four main steps. In Step 1, we find the unique solution on $(-\infty, u_\gamma]$, starting from the boundary condition at u_γ, and determine the first point $\tau^+_{\gamma,1}$ where the solution reaches y_1, i.e., the first point at which $k^\star(w, \cdot)$ changes; see (EC.45). In Step 2, we use y_1 as the initial value and find the unique solution on $(-\infty, \tau^+_{\gamma,1}]$, and determine the point $\tau^+_{\gamma,2}$ where the solution reaches the y_2, i.e., the second point at which $k^\star(w, \cdot)$ changes. This process continues iteratively, where the y_j value associated with the previous interval provides the initial value for the next, until we obtain solutions on $(-\infty, \tau^+_{\gamma,j}]$ for $j = 1, \ldots, b-1$, where $b \in \mathbb{N}$ is the number of breakpoints of $\mathbf{A}$; see Appendix EC.2.2.2. In Step 3, we use y_b as the initial value and find the unique solution on $(-\infty, \tau^+_{\gamma,b}]$. In Step 4, we combine the solutions from Steps 1–3 to obtain a continuously differentiable function on $[w_1, u_\gamma]$. After constructing this function, we establish its key properties and show that it uniquely solves the IVP (EC.67)–(EC.68).

Step 1: Solution on $(-\infty, u_\gamma]$. We construct a solution on the interval $(-\infty, u_\gamma]$ by considering the following IVP:

$$v'(w) = \frac{m'H^{-1}m}{2\sigma^2} v^2(w) - \frac{2\mu}{\sigma^2} v(w) - \frac{2}{\sigma^2} \phi^+_0(w, v(w)) + \frac{2\gamma}{\sigma^2}, \quad w \in (-\infty, u_\gamma], \tag{EC.70}$$

$$v(u_\gamma) = \kappa, \tag{EC.71}$$

where

$$\phi^+_0(w, y) \coloneqq \big[\alpha_{k_0} + \ell_{k_0}\big(d_{k_0} - m_{k_0} y\big)\big](w - w_1)/m_{k_0}, \quad (w, y) \in (-\infty, u_\gamma] \times \mathbb{R}, \tag{EC.72}$$

is the function that coincides with $\phi(w, y)$ for $w \in [w_1, \infty)$ and $y \in (y_1, \kappa]$. We next apply the results from Appendix EC.2.1 to obtain a solution to (EC.70)–(EC.71). To be specific, consider the Riccati equation (EC.29)–(EC.30) with the following constants:

$$c_0 = -\frac{2}{\sigma^2}\Big(\frac{\alpha_{k_0} + \ell_{k_0} d_{k_0}}{m_{k_0}} + \gamma\Big), \quad c_1 = -\frac{2\,(\alpha_{k_0} + \ell_{k_0} d_{k_0})}{\sigma^2 m_{k_0}},$$
$$c_2 = \frac{2}{\sigma^2}\big(\mu + \ell_{k_0}(w_1 + u_\gamma)\big), \quad c_3 = \frac{2\ell_{k_0}}{\sigma^2}, \quad c_4 = -\frac{m'H^{-1}m}{2\sigma^2}, \quad \text{and} \quad y_0 = \kappa.$$

By Lemma EC.4, there exists a unique solution $y^+_{\gamma,0} \in C^1[0, \infty)$ to this Riccati equation. It then follows that the function $\hat{v}^+_{\gamma,0} \in C^1(-\infty, u_\gamma]$ given by

$$\hat{v}^+_{\gamma,0}(w) \coloneqq y^+_{\gamma,0}(u_\gamma - w), \quad w \in (-\infty, u_\gamma], \tag{EC.73}$$

is the unique solution to (EC.70)–(EC.71). To construct the solution on the next interval, define $\tau^+_{\gamma,1} \coloneqq \sup\{w \le u_\gamma : \hat{v}^+_{\gamma,0}(w) \le y_1\}$, and note that by continuity of $\hat{v}^+_{\gamma,0}$, we have $\hat{v}^+_{\gamma,0}(\tau^+_{\gamma,1}) = y_1$. For notational convenience in the next step, define $\tau^+_{\gamma,0} \coloneqq u_\gamma$.

Step 2: Solution on $(-\infty, \tau^+_{\gamma,j}]$ **for** $j = 1, \ldots, b-1$**.** Given the solution $\hat{v}^+_{\gamma,j-1}$ on the interval $(-\infty, \tau^+_{\gamma,j-1}]$ for $j = 1, \ldots, b-1$, we construct a solution on the interval $(-\infty, \tau^+_{\gamma,j}]$ by considering the following IVP:

$$v'(w) = \frac{m'H^{-1}m}{2\sigma^2} v^2(w) - \frac{2\mu}{\sigma^2} v(w) - \frac{2}{\sigma^2} \phi^+_j(w, v(w)) + \frac{2\gamma}{\sigma^2}, \quad w \in (-\infty, \tau^+_{\gamma,j}], \tag{EC.74}$$

$$v(\tau^+_{\gamma,j}) = y_j, \tag{EC.75}$$

where

$$\phi^+_j(w, y) \coloneqq \left[\alpha_{k_j} + \ell_{k_j}\left(d_{k_j} - m_{k_j} y\right)\right](w - w_1)/m_{k_j}, \quad (w, y) \in (-\infty, \tau^+_{\gamma,j}] \times \mathbb{R}, \tag{EC.76}$$

is the function that coincides with $\phi(w, y)$ for $w \in [w_1, \infty)$ and $y \in (y_{j+1}, y_j]$. As in Step 1, we apply the results from Appendix EC.2.1 to obtain a solution. To be specific, consider the Riccati equation (EC.29)–(EC.30) with the following constants:

$$c_0 = -\frac{2}{\sigma^2}\left(\frac{\alpha_{k_j} + \ell_{k_j} d_{k_j}}{m_{k_j}} + \gamma\right), \quad c_1 = -\frac{2\,(\alpha_{k_j} + \ell_{k_j} d_{k_j})}{\sigma^2 m_{k_j}},$$
$$c_2 = \frac{2}{\sigma^2}\left(\mu + \ell_{k_j}(w_1 + \tau^+_{\gamma,j})\right), \quad c_3 = \frac{2\ell_{k_j}}{\sigma^2}, \quad c_4 = -\frac{m'H^{-1}m}{2\sigma^2}, \quad \text{and} \quad y_0 = y_j.$$

By Lemma EC.4, there exists a unique solution $y^+_{\gamma,j} \in C^1[0, \infty)$ to this Riccati equation. It then follows that the function $\hat{v}^+_{\gamma,j} \in C^1(-\infty, \tau^+_{\gamma,j}]$ given by

$$\hat{v}^+_{\gamma,j}(w) \coloneqq y^+_{\gamma,j}(\tau^+_{\gamma,j} - w), \quad w \in (-\infty, \tau^+_{\gamma,j}], \tag{EC.77}$$

is the unique solution to (EC.74)–(EC.75). To construct the solution on the next interval, define $\tau^+_{\gamma,j+1} \coloneqq \sup\{w \le \tau^+_{\gamma,j} : \hat{v}^+_{\gamma,j}(w) \le y_{j+1}\}$, and note that by continuity of $\hat{v}^+_{\gamma,j}$, we have $\hat{v}^+_{\gamma,j}(\tau^+_{\gamma,j+1}) = y_{j+1}$.

Step 3: Solution on $(-\infty, \tau^+_{\gamma,b}]$**.** Given the solution $\hat{v}^+_{\gamma,b-1}$ on the interval $(-\infty, \tau^+_{\gamma,b-1}]$, we construct a solution on the interval $(-\infty, \tau^+_{\gamma,b}]$ by considering the following IVP:

$$v'(w) = \frac{m'H^{-1}m}{2\sigma^2} v^2(w) - \frac{2\mu}{\sigma^2} v(w) - \frac{2}{\sigma^2} \phi^+_b(w, v(w)) + \frac{2\gamma}{\sigma^2}, \quad w \in (-\infty, \tau^+_{\gamma,b}], \tag{EC.78}$$

$$v(\tau^+_{\gamma,b}) = y_b, \tag{EC.79}$$

where

$$\phi^+_b(w, y) \coloneqq \alpha_{k_b}(w - w_1)/m_{k_b}, \quad (w, y) \in (-\infty, \tau^+_{\gamma,b}] \times \mathbb{R}, \tag{EC.80}$$

is the function that coincides with $\phi(w,y)$ for $w \in [w_1,\infty)$ and $y \in (-\infty, y_b]$. As in Steps 1–2, we apply the results from Appendix EC.2.1 to obtain a solution. To be specific, consider the Riccati equation (EC.29)–(EC.30) with the following constants:

$$c_0 = -\frac{2}{\sigma^2}\Big(\gamma + w_1 - \frac{\alpha_{k_b}\tau^+_{\gamma,k_b}}{m_{k_b}}\Big), \quad c_1 = -\frac{2\alpha_{k_b}}{\sigma^2 m_{k_b}},$$
$$c_2 = \frac{2\mu}{\sigma^2}, \quad c_3 = 0, \quad c_4 = -\frac{m'H^{-1}m}{2\sigma^2}, \quad \text{and} \quad y_0 = y_b.$$

By Lemma EC.3, there exists a unique solution $y^+_{\gamma,b} \in C^1[0,\infty)$ to this Riccati equation. It then follows that the function $\hat{v}^+_{\gamma,b} \in C^1(-\infty, \tau^+_{\gamma,b}]$ given by

$$\hat{v}^+_{\gamma,b}(w) \coloneqq y^+_{\gamma,b}(\tau^+_{\gamma,b} - w), \quad w \in (-\infty, \tau^+_{\gamma,b}], \tag{EC.81}$$

is the unique solution to (EC.78)–(EC.79).

Step 4: Solution on $[w_1, u_\gamma]$**.** For each $\gamma \geq 0$, we define the function $v^+_\gamma : [w_1, u_\gamma] \to \mathbb{R}$ by pasting together the solutions from Steps 1–3 as follows:

$$v^+_\gamma(w) \coloneqq \begin{cases} \hat{v}^+_{\gamma,j}(w), & w \in \big(\tau^+_{\gamma,j+1}, \tau^+_{\gamma,j}\big], \quad j = 0, 1, \ldots, B_\gamma - 1, \\ \hat{v}^+_{\gamma,B_\gamma}(w), & w \in \big[w_1, \tau^+_{\gamma,B_\gamma}\big], \end{cases} \tag{EC.82}$$

where $B_\gamma \coloneqq \max\{j = 1, \ldots, b : \tau^+_{\gamma,j} > w_1\}$ and the functions $\hat{v}^+_{\gamma,j}$ for $j = 0, 1, \ldots, B_\gamma$ are given by (EC.73), (EC.77), and (EC.81).

The next two results show that v^+_γ is continuously differentiable and strictly increasing, which together ensure that v^+_γ solves the IVP (EC.67)–(EC.68).

LEMMA EC.15. *For each $\gamma \geq 0$, the function $v^+_\gamma : [w_1, u_\gamma] \to \mathbb{R}$ defined by (EC.82) is continuously differentiable.*

LEMMA EC.16. *For each $\gamma \geq 0$, the function $v^+_\gamma \in C^1[w_1, u_\gamma]$ satisfies $(v^+_\gamma)'(u_\gamma) = 0$ and $(v^+_\gamma)'(w) > 0$ for all $w \in [w_1, u_\gamma)$. Consequently, v^+_γ is strictly increasing for each $\gamma \geq 0$.*

LEMMA EC.17. *For each $\gamma \geq 0$, the function $v^+_\gamma \in C^1[w_1, u_\gamma]$ is the unique solution to (EC.67)–(EC.68).*

Next, we establish several structural properties of v^+_γ that are essential in proving the existence and uniqueness of a solution to the Bellman equation (59)–(60).

LEMMA EC.18. *The mapping $\gamma \mapsto v^+_\gamma(w)$ is strictly decreasing in the following sense: For $0 \leq \gamma_1 < \gamma_2$, we have that $v^+_{\gamma_2}(w) < v^+_{\gamma_1}(w)$ for all $w \in [w_1, u_{\gamma_1}]$.*

LEMMA EC.19. *The mapping $\gamma \mapsto v^+_\gamma(w_1)$ is strictly decreasing with $v^+_0(w_1) \in (0, \kappa)$.*

LEMMA EC.20. *The mapping $\gamma \mapsto v^+_\gamma(w_1)$ is continuous on $[0,\infty)$.*

EC.2.2.4. Existence and Uniqueness of the Solution to the Bellman Equation. In this section, we use the results from Appendix EC.2.2.3 to prove the existence and uniqueness of a solution to the Bellman equation (59)–(60). Specifically, we construct a family of candidate solutions by pasting together the solutions to the IVPs (EC.47)–(EC.48) and (EC.67)–(EC.68), derived in Appendices EC.2.2.3.1 and EC.2.2.3.2, respectively. We then show that a function from this family uniquely solves the Bellman equation. To that end, for $\gamma \in [0, \gamma_\kappa]$, we define the function $v_\gamma : [l_\gamma, u_\gamma] \to \mathbb{R}$ as follows:

$$v_\gamma(w) := \begin{cases} v_\gamma^-(w), & w \in [l_\gamma, w_1], \\ v_\gamma^+(w), & w \in (w_1, u_\gamma], \end{cases} \tag{EC.83}$$

where $v_\gamma^- \in C^1[l_\gamma, w_1]$ is the unique solution to (EC.47)–(EC.48) (see Lemma EC.14) and $v_\gamma^+ \in C^1[w_1, u_\gamma]$ is the unique solution to (EC.67)–(EC.68) (see Lemma EC.17). The following result proves that there exists a unique γ for which v_γ is continuously differentiable.

LEMMA EC.21. *There exists a unique $\gamma^\star \in [0, \gamma_\kappa]$ such that $v_{\gamma^\star} \in C^1[l_{\gamma^\star}, u_{\gamma^\star}]$. Furthermore, $\gamma^\star > 0$.*

The remaining results now show that there exists a unique solution to the Bellman equation (59)–(60).

COROLLARY EC.4. *The tuple $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ is a solution to the Bellman equation (59)–(60). Furthermore, $l_{\gamma^\star} < w_0 \le w_1 < u_{\gamma^\star}$ and the solution $v_{\gamma^\star} \in C^1[l_{\gamma^\star}, u_{\gamma^\star}]$ is nonnegative and strictly increasing.*

LEMMA EC.22. *If (l, u, γ, v) is a solution to the Bellman equation (59)–(60), then $l < w_0$, $u > w_1$, and $\gamma \in [0, \gamma_\kappa]$. Moreover, the endpoints are uniquely determined as $u = u_\gamma$ and $l = l_\gamma$.*

COROLLARY EC.5. *The solution to the Bellman equation (59)–(60) is unique.*

EC.3. Proofs of Main Results

Proof of Proposition 1. Fix n and let (λ^n, T^n, R^n) be an arbitrary admissible policy for the nth system. Then, by (13), the cumulative profit process for the nth system is given by

$$\begin{aligned} V^n(t) := \int_0^t \Pi^n\big(\lambda^n(s)\big)\, ds - \sum_{k \in \mathcal{S}} \int_0^t \upsilon_k^n\big(w_k^n(s) - \delta_k^n\big)\, dA_k^n(s) - \sum_{k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k^n \big[Q_k^n(s)\big]^- ds \\ - \sum_{k \in \mathcal{S}_{\mathrm{w}}} d_k^n M_k\Big(\int_0^t \ell_k^n \big[Q_k^n(s)\big]^+ ds\Big) - \sum_{k \in \mathcal{S}} r_k^n R_k^n(t). \end{aligned} \tag{EC.84}$$

Since each of the last four terms on the right-hand side of (EC.84) is nonpositive, it follows that

$$V^n(t) \le \int_0^t \Pi^n\big(\lambda^n(s)\big)\, ds = n \int_0^t \Pi\big(n^{-1}\lambda^n(s)\big)\, ds \quad \text{for all} \quad t \ge 0. \tag{EC.85}$$

where the equality follows from the second equality in (17). Moreover, by (19), we have $n^{-1}\lambda^n(s) = \lambda^\star + n^{-1/2}\zeta(s)$, implying that $n^{-1}\lambda^n(s) \in \mathcal{L}$ for all n sufficiently large. Since $\Pi(\lambda) \le \Pi(\lambda^\star)$ for all $\lambda \in \mathcal{L}$ by Assumption 3, it follows from (EC.85) that

$$V^n(t) \le n\int_0^t \Pi(\lambda^\star)\,ds = n\,\Pi(\lambda^\star)\,t \quad \text{for all} \quad t \ge 0,$$

for n sufficiently large. □

PROPOSITION EC.1. *For every admissible policy (ζ, Y, O) for the BCP (37), there exists an admissible policy (L, U, z, θ) for the EWF (49), and its cost is less than or equal to that of the policy (ζ, Y, O) for the BCP (37). Conversely, for every admissible policy (L, U, z, θ) for the EWF (49), there exists an admissible policy (ζ, Y, O) for the BCP (37), and these two policies have the same cost.*

Proof of Proposition EC.1.[7] Let (ζ, Y, O) be an admissible policy for the BCP (37), with the corresponding state descriptor $Z = \{Z(t) : t \ge 0\}$ given by (28)–(30). We will show that there exists an admissible policy (L, U, z, θ) for the EWF (49) with a long-run average cost less than or equal to that of the policy (ζ, Y, O) for the BCP (37). Define the processes

$$L(t) := \sum_{k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} Y_k(t), \quad U(t) := \sum_{k \in \mathcal{S}} m_k O_k(t), \quad \text{and} \quad \theta(t) := \sum_{k \in \mathcal{S}} m_k \zeta_k(t), \quad t \ge 0. \tag{EC.86}$$

Next, define $z(t, w) := Z(t)$ for $t \ge 0$ and $w \in \mathbb{R}$ and $W(t) := m'Z(t)$ for $t \ge 0$. It then follows from (28)–(30) and (EC.86) that

$$W(t) = B(t) + \int_0^t \theta(s)\,ds - \int_0^t \sum_{k \in \mathcal{S}_{\mathrm{w}}} m_k \ell_k z_k^+(s, W(s))\,ds + L(t) - U(t), \quad t \ge 0. \tag{EC.87}$$

By (34) and (EC.87), it follows that (44)–(45) hold. Moreover, by (32)–(33) and (36), it follows that (47) and (46) hold. Hence, (L, U, z, θ) is an admissible policy for the EWF (49). Finally, by (41)–(43), observe that

$$c(\theta(t)) \le \zeta(t)' H \zeta(t) \quad \text{and} \quad \kappa\, U(t) \le \sum_{k \in \mathcal{S}} r_k O_k(t) \quad \text{for all} \quad t \ge 0.$$

It then follows from (35) and (49) that (L, U, z, θ) achieves a lower cost for the EWF (49) than (ζ, Y, O) does for the BCP (37).

Conversely, let (L, U, z, θ) be an admissible policy for the EWF (49), with the corresponding state descriptor $W = \{W(t) : t \ge 0\}$ given by (44). We will show that there exists an admissible policy (ζ, Y, O) for

[7] This proof requires the technical subtlety that $B(t) = \sum_{k \in \mathcal{S}} m_k X_k(t)$ almost surely for $t \ge 0$, where X_k for $k \in \mathcal{S}$ and B are the Brownian motions from Sections 4 and 5, respectively. Since B and $\sum_{k \in \mathcal{S}} m_k X_k$ are identically distributed, this equality can be achieved by potentially expanding the probability space on which they are defined; see Skorokhod's representation theorem. For a rigorous treatment of this issue, see Harrison and Williams (2005).

the BCP (37) with a long-run average cost equal to that of the policy (L, U, z, θ) for the EWF (49). Define the processes

$$O_k(t) := \begin{cases} U(t)/m_k, & k = k^\star, \\ 0, & k \neq k^\star, \end{cases} \quad \text{and} \quad \zeta(t) := \frac{H^{-1}m}{m'H^{-1}m}\,\theta(t), \quad t \geq 0. \tag{EC.88}$$

By (41) and Lemma 1, it follows that $m'\zeta(t) = \theta(t)$ for $t \geq 0$. Next, define the process $Z = \{Z(t) : t \geq 0\}$ as $Z_k(t) := z_k(t, W(t))$ for $k \in \mathcal{S}$ and $t \geq 0$. Moreover, define the process $Y = \{Y(t) : t \geq 0\}$ as follows:

$$Y_k(t) := m_k\Big(Z_k(t) - X_k(t) - \int_0^t \zeta_k(s)\,ds + \int_0^t \mathbf{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k Z_k^+(s)\,ds + O_k(t)\Big), \quad k \in \mathcal{S}^{\mathrm{MTO}}, \tag{EC.89}$$

$$Y_k(t) := m_k\Big(Z_k(t) - X_k(t) - \int_0^t \zeta_k(s)\,ds + \int_0^t \ell_k Z_k^+(s)\,ds + O_k(t) + \sum_{j\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \mu_j Y_j(t)\Big), \quad k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \tag{EC.90}$$

$$Y_k(t) := m_k\Big(Z_k(t) - X_k(t) - \int_0^t \zeta_k(s)\,ds + O_k(t)\Big), \quad k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}. \tag{EC.91}$$

It then readily follows from (47), (45), and (EC.89)–(EC.91) that (28)–(30), (34), and (36) hold. Moreover, by (44)–(45) and (EC.88)–(EC.91), we have that

$$L(t) = \sum_{k\in\mathcal{S}^{\mathrm{MTO}}\cup\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} Y_k(t), \quad t \geq 0. \tag{EC.92}$$

Therefore, by (46), (EC.88), and (EC.92), it follows that (32)–(33) hold. Hence, (ζ, Y, O) is an admissible policy for the BCP (37). Finally, by (35), (41), (49), (EC.88), and Lemma 1, the policy (ζ, Y, O) achieves the same cost for the BCP (37) as (L, U, z, θ) does for the EWF (49). □

Proof of Proposition 2. Fix a barrier policy (L, U, z, θ) with a lower barrier at l and an upper barrier at u, and assume that $\gamma \in \mathbb{R}$ and $f \in C^2[l, u]$ jointly satisfy (52)–(53). From (46) and Harrison (2013, Appendix B.2), the control processes L and U are of finite variation almost surely. Using this fact and applying Itô's Lemma to the workload process in (39) yields

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(s))\,dW(s) + \int_0^t \frac{1}{2}\sigma^2 f''(W(s))\,ds, \quad t \geq 0. \tag{EC.93}$$

Then, by (39), (51), and (EC.93), it follows that

$$\begin{aligned} f(W(t)) - f(W(0)) = &\int_0^t f'(W(s))\,d(B(s) - \mu s) + \int_0^t \Gamma_{z,\theta} f(W(s))\,ds \\ &+ \int_0^t f'(W(s))\,dL(s) - \int_0^t f'(W(s))\,dU(s), \quad t \geq 0. \end{aligned} \tag{EC.94}$$

Recall that by Definition 1, the control processes L and U increase only when $W(t) = l$ and $W(t) = u$, respectively. Thus, it follows from (53) that

$$\int_0^t f'(W(s))\,dL(s) = f'(l)L(t) = 0 \quad \text{and} \quad \int_0^t f'(W(s))\,dU(s) = f'(u)U(t) = \kappa U(t), \quad t \geq 0. \tag{EC.95}$$

Substituting (EC.95) into (EC.94) and taking expectations of both sides yields

$$\begin{aligned}\mathbb{E}\big[f(W(t))-f(W(0))\big] &= \mathbb{E}\Big[\int_0^t f'(W(s))\,d(B(s)-\mu s)\Big] + \mathbb{E}\Big[\int_0^t \Gamma_{z,\theta} f(W(s))\,ds\Big] - \mathbb{E}\Big[\int_0^t \kappa\,dU(s)\Big] \\ &= \mathbb{E}\Big[\int_0^t \Gamma_{z,\theta} f(W(s))\,ds\Big] - \mathbb{E}\Big[\int_0^t \kappa\,dU(s)\Big], \qquad \text{(EC.96)}\end{aligned}$$

where the second equality holds by Harrison (2013, Proposition 4.7) since

$$\mathbb{E}\Big[\int_0^t \big[f'(W(s))\big]^2\,ds\Big] \le \int_0^t \sup_{w\in[l,u]} \big[f'(w)\big]^2\,ds < \infty,$$

which is finite since f' is a continuous function over the compact set $[l,u]$. Dividing both sides of (EC.96) by t, using (52), and rearranging terms gives

$$\begin{aligned}\frac{1}{t}\,\mathbb{E}\big[f(W(t))-f(W(0))\big] + \frac{1}{t}\,\mathbb{E}\Big[\int_0^t c\big(\theta(W(s))\big)\,ds + \sum_{k\in\mathcal{S}}\int_0^t \upsilon_k\big(z_k(W(s))-\lambda_k^\star\delta_k\big)\,ds \\ + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}}\int_0^t h_k z_k^-(W(s))\,ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}}\int_0^t d_k\ell_k z_k^+(W(s))\,ds + \kappa U(t)\Big] = \gamma. \qquad \text{(EC.97)}\end{aligned}$$

Finally, we have $\lim_{t\to\infty} t^{-1}\,\mathbb{E}[f(W(t))-f(W(0))]=0$, since f is continuous and $W(t)\in[l,u]$ for all $t\ge 0$ by Definition 1. Using this fact and taking the limit as $t\to\infty$ in (EC.97) yields the desired result. □

Proof of Theorem 1. Theorem 1 follows directly from Corollaries EC.4 and EC.5 in Appendix EC.2.2.4. Indeed, Appendix EC.2.2.4 provides an explicit characterization of the solution to the Bellman equation (59)–(60). This is accomplished by establishing several auxiliary lemmas in Appendix EC.2.2.3 on two related initial value problems. The solution to the Bellman equation is ultimately constructed by smoothly pasting together the solutions to these initial value problems. □

Proof of Corollary 1. Let $(l^\star,u^\star,\gamma^\star,v^\star)$ be the unique solution to the Bellman equation (59)–(60) as guaranteed by Theorem 1. By the fundamental theorem of calculus and the definition of $f^\star\in C^2[l^\star,u^\star]$ in (62), we have that

$$(f^\star)'(w)=v^\star(w) \quad\text{and}\quad (f^\star)''(w)=(v^\star)'(w) \quad \text{for all} \quad w\in[l^\star,u^\star].$$

Since $(l^\star,u^\star,\gamma^\star,v^\star)$ uniquely satisfies (59), by replacing $(f^\star)'$ with $v^\star$ and $(f^\star)''$ with $(v^\star)'$, it immediately follows that $(l^\star,u^\star,\gamma^\star,f^\star)$ uniquely satisfies (55), up to an additive constant in $f^\star$. Furthermore, from the boundary conditions (60), we have that

$$(f^\star)'(l^\star)=v^\star(l^\star)=0,\quad (f^\star)''(l^\star)=(v^\star)'(l^\star)=0,\quad (f^\star)''(u^\star)=(v^\star)'(u^\star)=0,\quad (f^\star)'(u^\star)=v^\star(u^\star)=\kappa,$$

which immediately implies that $(l^\star,u^\star,\gamma^\star,f^\star)$ satisfies (56). We conclude that $(l^\star,u^\star,\gamma^\star,f^\star)$ is the unique solution to the Bellman equation (55)–(56), up to an additive constant in $f^\star$. Finally, since $v^\star$ is strictly increasing by Theorem 1, it follows from (62) that $f^\star$ is strictly increasing and strictly convex. □

Proof of Theorem 2. Let $(l^\star, u^\star, \gamma^\star, v^\star)$ be the unique solution to the Bellman equation (59)–(60) guaranteed by Theorem 1, and let $f^\star \in C^2[l^\star, u^\star]$ be the function defined in (62). Consider the barrier policy $(L^\star, U^\star, z^\star, \theta^\star)$ with a lower barrier at $l^\star$, an upper barrier at $u^\star$, effective drift rate function $\theta^\star$ defined by (63), and workload configuration function $z^\star$ defined by (64). By (57), (63)–(64), and Corollary 1, it follows that $f^\star$ satisfies (52)–(53) with $z = z^\star$, $\theta = \theta^\star$, and $\gamma = \gamma^\star$. Therefore, by Proposition 2, the barrier policy $(L^\star, U^\star, z^\star, \theta^\star)$ achieves a long-run average expected cost of $\gamma^\star$ for the EWF (49). To complete the proof, it remains to show that $\gamma^\star$ is a lower bound on the long-run average expected cost of any admissible policy (L, U, z, θ) to the EWF (49). The remainder of the proof consists of two parts. In the first part, we extend the solution of the Bellman equation to the entire real line and establish an inequality involving the constant $\gamma^\star$. In the second part, we use the estimate from the first part, along with Itô's Lemma, to prove that no admissible policy can achieve a long-run average expected cost lower than $\gamma^\star$.

We begin the first part of the proof by extending the solution of the Bellman equation to the entire real line. (This is necessary because an arbitrary admissible policy may push the workload process beyond the lower barrier $l^\star$ or the upper barrier $u^\star$.) To that end, we extend $f^\star$ to the entire real line as follows:

$$f^\star(w) := \begin{cases} 0, & w \in (-\infty, l^\star), \\ \displaystyle\int_{l^\star}^{w} v^\star(x)\,dx, & w \in [l^\star, u^\star], \\ \displaystyle\int_{l^\star}^{u^\star} v^\star(x)\,dx + \kappa(w - u^\star), & w \in (u^\star, \infty). \end{cases} \tag{EC.98}$$

We now verify some properties of this extended function. First, since $v^\star \in C^1[l^\star, u^\star]$ with $v^\star(l^\star) = 0$ and $v^\star(u^\star) = \kappa$, it follows from (EC.98) that $f^\star \in C^2(\mathbb{R})$. Second, we have $(f^\star)' \equiv 0$ on $(-\infty, l^\star]$, $(f^\star)' \equiv v$ on $[l^\star, u^\star]$, and $(f^\star)' \equiv \kappa$ on $[u^\star, \infty)$. Therefore, since $v^\star$ is strictly increasing on $[l^\star, u^\star]$ with $v^\star(l^\star) = 0$ and $v^\star(u^\star) = \kappa$ (by Theorem 1), it follows that $0 \le (f^\star)'(w) \le \kappa$ for all $w \in \mathbb{R}$. Next, we claim that

$$\min_{z \in \mathcal{A}(w),\, x \in \mathbb{R}} \left\{ \frac{1}{2}\sigma^2 (f^\star)''(w) + (\mu + x)\,(f^\star)'(w) + c(x) + \varphi\big(z, (f^\star)'(w)\big) \right\} \ge \gamma^\star, \quad w \in \mathbb{R}. \tag{EC.99}$$

We consider the sets $[l^\star, u^\star]$ and $\mathbb{R} \setminus [l^\star, u^\star]$ separately. On the one hand, for $w \in [l^\star, u^\star]$, (EC.99) holds with equality since $(l^\star, u^\star, \gamma^\star, f^\star)$ satisfies (55)–(56) by Corollary 1. On the other hand, for $w \in \mathbb{R} \setminus [l^\star, u^\star]$, it follows from Lemma EC.7 and the properties of the extended function $f^\star$ that (EC.99) holds with a strict inequality.[8] This completes the first part of the proof.

The second part of the proof uses the results from the first part to show that our proposed policy is optimal for the workload formulation. Specifically, we show that no admissible policy can achieve an objective lower

[8] In particular, (EC.98) implies that $(f^\star)'(w) = (f^\star)'(l^\star) = 0$ and $(f^\star)''(w) = (f^\star)''(l^\star) = 0$ for all $w \in (-\infty, l^\star)$. Similarly, $(f^\star)'(w) = (f^\star)'(u^\star) = \kappa$ and $(f^\star)''(w) = (f^\star)''(u^\star) = 0$ for all $w \in (u^\star, \infty)$. Then, by Lemma EC.7, observe that $\phi(w, (f^\star)'(w)) = \phi(w, 0) > \phi(l^\star, 0)$ for all $w \in (-\infty, l^\star)$, and $\phi(w, (f^\star)'(w)) = \phi(w, \kappa) > \phi(u^\star, \kappa)$ for all $w \in (u^\star, \infty)$. Combining these inequalities with the fact that (EC.99) holds with equality at $l^\star$ and $u^\star$ establishes that (EC.99) holds with a strict inequality for all $w \in \mathbb{R} \setminus [l^\star, u^\star]$.

than $\gamma^\star$. To that end, let (L, U, z, θ) be an arbitrary admissible policy for the EWF (49), and let W be the resulting workload process defined in (44). By Itô's Lemma, for $t \geq 0$ we have that

$$\mathbb{E}\Big[f^\star(W(t)) - f^\star(W(0))\Big] = \mathbb{E}\Big[\int_0^t \Gamma_{z,\theta} f^\star(W(s))\,ds + \int_0^t (f^\star)'(W(s))\,d\big(L(s) - U(s)\big)\Big], \quad \text{(EC.100)}$$

where the differential operator $\Gamma_{z,\theta}$ is given by (51). Furthermore, by (EC.99), we have that

$$\frac{1}{2}\sigma^2 (f^\star)''\big(W(t)\big) + \big(\mu + \theta(W(t))\,(f^\star)'(W(t)\big) + c\big(\theta(W(t))\big) + \varphi\big(z(W(t)), (f^\star)'(W(t))\big) \geq \gamma^\star.$$

Integrating both sides of this inequality over $[0, t]$, taking expectations, and rearranging terms yields

$$\mathbb{E}\Big[\int_0^t \Gamma_{z,\theta} f^\star(W(s))\,ds\Big] \geq \gamma^\star t - \mathbb{E}\Big[\int_0^t c\big(\theta(W(s))\big)\,ds + \sum_{k\in\mathcal{S}} \int_0^t \upsilon_k\big(z_k(W(s)) - \lambda_k^\star \delta_k\big)\,ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k z_k^-(W(s))\,ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}} \int_0^t d_k \ell_k z_k^+(W(s))\,ds\Big].$$

Moreover, since L and U are nondecreasing (see (46)) and $0 \leq (f^\star)'(w) \leq \kappa$ for all $w \in \mathbb{R}$, we have that

$$\mathbb{E}\Big[\int_0^t (f^\star)'(W(s))\,dL(s)\Big] \geq 0 \quad \text{and} \quad \mathbb{E}\Big[\int_0^t (f^\star)'(W(s))\,dU(s)\Big] \leq \mathbb{E}\big[\kappa U(t)\big].$$

Substituting the two preceding displays into (EC.100) then yields

$$\mathbb{E}\Big[f^\star(W(t)) - f^\star(W(0))\Big] \geq \gamma^\star t - \mathbb{E}\Big[\kappa U(t) + \int_0^t c\big(\theta(W(s))\big)\,ds + \sum_{k\in\mathcal{S}} \int_0^t \upsilon_k\big(z_k(W(s)) - \lambda_k^\star \delta_k\big)\,ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k z_k^-(W(s))\,ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}} \int_0^t d_k \ell_k z_k^+(W(s))\,ds\Big]. \quad \text{(EC.101)}$$

Finally, since $(f^\star)'$ is bounded, we have that

$$\limsup_{t\to\infty} \frac{\mathbb{E}\big[f^\star(W(t)) - f^\star(W(0))\big]}{t} \leq \limsup_{t\to\infty} \frac{\sup_{w\in\mathbb{R}} (f^\star)'(w)\cdot \mathbb{E}\big[|W(t) - W(0)|\big]}{t} = 0,$$

where the inequality follows from Jensen's inequality and the mean value theorem, and the equality follows from (47). The desired result then follows by dividing both sides of (EC.101) by t, taking the lim sup as $t \to \infty$, and applying the limiting bound established above on the left-hand side. This completes the second part of the proof. □

COROLLARY EC.6. *There exists an optimal control policy $(\zeta^\star, Y^\star, O^\star)$ to the BCP (37) with $\zeta^\star(t) = -(H^{-1}m/2)\,v^\star(m'Z(t))$ for $t \geq 0$, and it has a long-run average expected cost of $\gamma^\star$.*

Proof of Corollary EC.6. By Theorem 2, there exists an optimal policy $(L^\star, U^\star, z^\star, \theta^\star)$ for the EWF (49) with a long-run average expected cost of $\gamma^\star$. Therefore, by Proposition EC.1, there exists an admissible policy $(\zeta^\star, Y^\star, O^\star)$ for the BCP (37) with a long-run average expected cost of $\gamma^\star$. Conversely, by Proposition EC.1, for every admissible policy (ζ, Y, O) for the BCP (37), there exists an admissible policy (L, U, z, θ) for the EWF (49) with a long-run average expected cost that is less than or equal to that of (ζ, Y, O) for the BCP (37). Hence, $(\zeta^\star, Y^\star, O^\star)$ is an optimal solution for the BCP (37) with a long-run average expected cost of $\gamma^\star$. This completes the proof. □

EC.4. Proofs of Results in Appendix EC.2

Proof of Lemma EC.1. We first establish the existence of a solution to (EC.29)–(EC.30). This result is included here for completeness, as it previews the closed-form solution derived later. We consider two cases: $c_3 = 0$ and $c_3 \neq 0$. In both cases, the solution can be expressed in terms of special functions; see, e.g., Zaitsev and Polyanin (2002, Sections 1.2.2-2 and 2.1.2-3). If $c_3 = 0$, the solution can be expressed in terms of Airy functions. If $c_3 \neq 0$, the solution can be expressed in terms of confluent hypergeometric functions.

We next establish the uniqueness of a solution to (EC.29)–(EC.30). Observe that (EC.29) can be written in the form $y'(x) = f(x, y(x))$, where $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$f(x, y) \coloneqq c_4 y^2 + (c_3 x + c_2) y + c_1 x + c_0, \quad (x, y) \in [0, \infty) \times \mathbb{R}.$$

By the Picard–Lindelöf Theorem, uniqueness follows if f is locally Lipschitz in y, i.e., if f is Lipschitz in y when restricted to the compact set $[0, N] \times [-M, M]$ for arbitrary $N, M > 0$. To see this, note that for $x \in [0, N]$ and $y_1, y_2 \in [-M, M]$,

$$\begin{aligned}
\left|f(x, y_2) - f(x, y_1)\right| &= \left|\left(c_4 y_2^2 + (c_3 x + c_2) y_2\right) - \left(c_4 y_1^2 + (c_3 x + c_2) y_1\right)\right| \\
&\le \left|c_4\right|\left|y_2^2 - y_1^2\right| + \left|c_3 x + c_2\right|\left|y_2 - y_1\right| \\
&= \left(\left|c_4\right|\left|y_2 + y_1\right| + \left|c_3 x + c_2\right|\right)\left|y_2 - y_1\right| \\
&\le \left(2M\left|c_4\right| + N\left|c_3\right| + \left|c_2\right|\right)\left|y_2 - y_1\right| \\
&= L_{N,M}\left|y_2 - y_1\right|,
\end{aligned}$$

where $L_{N,M} \coloneqq 2M\left|c_4\right| + N\left|c_3\right| + \left|c_2\right| < \infty$. Therefore, f is locally Lipschitz in y, and it follows that the solution to (EC.29)–(EC.30) is unique. □

Proof of Lemma EC.2. Apart from minor notational differences, the proof is identical to that of Lemma 23 in Alwan et al. (2024). □

Proof of Lemma EC.3. Apart from minor notational differences, the proof is identical to that of Lemma EC.2 in Ata and Barjesteh (2022). □

Proof of Lemma EC.4. It follows from Lemma EC.2, along with (EC.34)–(EC.35) and the surrounding discussion, that y satisfies (EC.29). Moreover, by the product rule and the chain rule for differentiation, for $x \in [0, \infty)$ it follows that

$$\begin{aligned}
z'(x) = C z(x) + \exp(Cx)\Bigg(& C_1 c_3 (x - B)\Phi'\big(A, \frac{1}{2}, \frac{c_3}{2}(x - B)^2\big) \\
&+ C_2\sqrt{\tfrac{c_3}{2}}\left(\Phi\big(A + \frac{1}{2}, \frac{3}{2}, \frac{c_3}{2}(x - B)^2\big) + c_3 (x - B)^2 \Phi'\big(A + \frac{1}{2}, \frac{3}{2}, \frac{c_3}{2}(x - B)^2\big)\right)\Bigg).
\end{aligned}$$

Using the expressions for C_1 and C_2 in , it is easily verified that $z(0) = 1$ and $z'(0) = -c_4 y_0$. We conclude that y satisfies (EC.30). Finally, uniqueness follows from Lemma EC.1. □

Proof of Lemma EC.5. By (43) and Assumption 2, we have $\kappa \le d_k/m_k$ for all $k \in \mathcal{S}_{\text{w}}$, implying that $d_k - m_k y \ge 0$ for all such k. Also recall that $\alpha_k > 0$ for all $k \in \mathcal{S}$, $\ell_k \ge 0$ for all $k \in \mathcal{S}_{\text{w}}$, $\beta_k = 0$ for $k \in \mathcal{S}_{\text{w}}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}$, $\beta_k > 0$ for $k \in \mathcal{S}_{\text{o}}^{\text{MTO}}$, and $h_k > 0$ for $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ (see Sections 3.4 and 4.2). Substituting these inequalities into the definitions of $\hat{\alpha}_k(y)$ and $\hat{\beta}_k$ in (65), it follows that $\hat{\alpha}_k(y) > 0$ for all $k \in \mathcal{S}$ and $y \in (-\infty, \kappa]$, $\hat{\beta}_k > 0$ for all $k \in \mathcal{S}_{\text{o}}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$, and $\hat{\beta}_k = 0$ for all $k \in \mathcal{S}_{\text{w}}^{\text{MTO}} \cup \mathcal{S}_{\text{o}}^{\text{MTS}}$. □

Proof of Corollary EC.1. By (66)–(67) and Lemma EC.5, we have $\varphi(z, y) \ge 0$ for all $z \in \mathbb{R}^K$ and $y \in (-\infty, \kappa]$. It then follows from (58) that $\phi(w, y) \ge 0$ for all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$. □

Proof of Lemma EC.6. We must show that $\mathcal{Z}(w, y) \in \arg\min_{z \in \mathcal{A}(w)} \varphi(z, y)$ for all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$. To do so, we first verify that $\mathcal{Z}(w, y) \in \mathcal{A}(w)$ for all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$. For example, when $w \in (w_1, \infty)$, we have $m'\mathcal{Z}(w, y) = \sum_{k \in \mathcal{S}} \lambda_k^\star \delta_k m_k + (w - w_1) = w$, where the second equality follows from the fact that $w_1 = \sum_{k \in \mathcal{S}_{\text{o}}} \lambda_k^\star \delta_k m_k = \sum_{k \in \mathcal{S}} \lambda_k^\star \delta_k m_k$, which holds since $\delta_k = 0$ for $k \in \mathcal{S}_{\text{w}}$. The cases of $w \in (-\infty, w_0)$ and $w \in [w_0, w_1]$ follow similarly and are therefore omitted.

To complete the proof, we must show that $\varphi(z, y) \ge \varphi(\mathcal{Z}(w, y), y)$ for all $w \in \mathbb{R}$, $y \in (-\infty, \kappa]$, and $z \in \mathcal{A}(w)$. We establish this inequality by considering w over the intervals $(-\infty, w_0)$, $[w_0, w_1]$, and (w_1, ∞). While we omit the full mathematical details, we provide sufficient explanation to guide the reader through the argument.

First, consider the case where $w \in [w_0, w_1]$. In this case, the workload configuration $\mathcal{Z}(w, y)$ allocates a workload of w_0 to the online MTO products without incurring earliness or tardiness costs. The remaining workload, $w - w_0$, is held in the online MTS products while ensuring that $\mathcal{Z}_k(w, y) \le \lambda_k^\star \delta_k$ for $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$. Consequently, no tardiness costs are incurred for the online MTO products. No workload is held in the walk-in classes, which ensures their earliness, tardiness, and abandonment costs are zero. We conclude that $\varphi(\mathcal{Z}(w, y), y) = 0$, and so the desired result follows from Corollary EC.1. Finally, since there are uncountably many ways to distribute workload among the online MTS products without incurring tardiness costs, the optimal workload configuration need not be unique.

Second, consider the case where $w \in (w_1, \infty)$. In this case, the workload configuration $\mathcal{Z}_k(w, y)$ holds a workload of w_1 in the online classes without incurring earliness or tardiness costs. The remaining workload, $w - w_1$, is then held in the cheapest manner, i.e., in the product class with the lowest (effective) tardiness and abandonment cost per unit of work.

Finally, consider the case where $w \in (-\infty, w_0)$. In this case, the workload configuration $\mathcal{Z}_k(w, y)$ initially allocates a workload of w_0 to the online MTO products, with all other products receiving zero workload. If left unchanged, this allocation would result in zero earliness, tardiness, holding, and abandonment costs across all product classes. However, since $w_0 - w > 0$, the workload configuration subtracts this excess workload in the cheapest manner by decreasing the workload of the online MTO and walk-in MTS products in increasing order of earliness or holding cost per unit of work. Meanwhile, the workload of the walk-in MTO and online MTS products remains zero. □

Proof of Corollary EC.2. By Lemma EC.6, it follows that $\phi(w,y) = \varphi(\mathcal{Z}(w,y),y)$ for all $w \in \mathbb{R}$ and $y \in (-\infty,\kappa]$. Then, substituting (EC.40)–(EC.42) into (54) and (58) and simplifying the terms yields the desired result. □

Proof of Lemma EC.7. Fix $y \in (-\infty,\kappa]$. We first establish part (a). It follows from (EC.38)–(EC.39) and Corollary EC.2 that the mapping $w \mapsto \phi(w,y)$ is piecewise linear over its entire domain. Thus, to prove continuity, it suffices to show that it is continuous at its breakpoints, i.e., the points where the derivative changes. It follows from (EC.38) and Corollary EC.2 that the breakpoints are $\tau_{I-1}^-, \ldots, \tau_1^-, \tau_0^-$, and w_1. It is straightforward to verify that $w \mapsto \phi(w,y)$ is left-continuous on its entire domain and that its left and right limits agree at each breakpoint. Therefore, $w \mapsto \phi(w,y)$ is continuous. Moreover, since it is piecewise linear, it is differentiable except at a set of Lebesgue measure zero, completing the proof of part (a).

We next establish part (b). On the interval $(-\infty,w_0)$, since $\hat{\beta}_{i^\star(k^\star(w,y))} > 0$ (by Lemma EC.5) and $i^\star(k^\star(w,y)) \in \mathcal{S}_{\text{o}}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$ (by the definition of the mapping $i^\star$), it follows from Corollary EC.2 and part (a) that $w \mapsto \phi(w,y)$ is a continuous and piecewise-linear function with negative slopes. Consequently, $w \mapsto \phi(w,y)$ is strictly decreasing on $(-\infty,w_0)$ with $\phi(w,y) \to \infty$ as $w \to -\infty$. On the interval $[w_0,w_1]$, it follows from Corollary EC.2 that $\phi(w,y) = 0$ for all $w \in [w_0,w_1]$, as desired. On the interval (w_1,∞), since $\hat{\alpha}_{k^\star(w,y)}(y) > 0$ (by Lemma EC.5) and $w \mapsto k^\star(w,y)$ remains constant over the entire interval (by (EC.37)), it follows from Corollary EC.2 and part (a) that $w \mapsto \phi(w,y)$ is a linear function with a constant positive slope. Consequently, $w \mapsto \phi(w,y)$ is strictly increasing on (w_1,∞) with $\phi(w,y) \to \infty$ as $w \to \infty$. □

Proof of Lemma EC.8. We begin by establishing that $\mathbf{A}$ is strictly positive, nonincreasing, continuous, concave, and non-differentiable at at most finitely many points. First, by Lemma EC.5, we have that $\mathbf{A}(y) > 0$ for $y \in (-\infty,\kappa]$, i.e., $\mathbf{A}$ is strictly positive. Second, by (65), the mapping $y \mapsto \hat{\alpha}_k(y)$ is nonincreasing and continuous for each $k \in \mathcal{S}$. Since the pointwise infimum of nonincreasing and continuous functions is nonincreasing and continuous, it follows that $\mathbf{A}$ is nonincreasing and continuous. Third, since the mappings $y \mapsto \hat{\alpha}_k(y)$ for $k \in \mathcal{S}$ are affine and the pointwise infimum of affine functions is concave (see, e.g., Boyd and Vandenberghe (2004, Chapter 3)), it follows that $\mathbf{A}$ is concave. Finally, since $\mathbf{A}$ is the pointwise minimum of (a finite number of) affine functions, it can only be non-differentiable at points where two of these functions intersect. Consider $k_1,k_2 \in \mathcal{S}$ such that $\ell_{k_1} < \ell_{k_2}$, and suppose that their corresponding affine functions intersect at some $y' \in (-\infty,\kappa]$, i.e., $\hat{\alpha}_{k_1}(y')/m_{k_1} = \hat{\alpha}_{k_2}(y')/m_{k_2}$. Since $\ell_{k_1} < \ell_{k_2}$, it follows that $\hat{\alpha}_{k_1}(y)/m_{k_1} < \hat{\alpha}_{k_2}(y)/m_{k_2}$ for all $y < y'$. This implies that each affine function is either never equal to the pointwise minimum or it is equal to the pointwise minimum on a convex interval. Since there are only finitely many such functions, $\mathbf{A}$ is non-differentiable at at most finitely many points.[9]

We now show that $\mathbf{A}$ is either constant over its entire domain or remains constant up to some point, after which it has a strictly negative derivative except at finitely many points. We do so by considering two exhaustive cases.

Case 1. Assume that there exists $k_0 \in \mathcal{S}^0$ such that $\alpha_{k_0}/m_{k_0} \le \hat{\alpha}_k(\kappa)/m_k$ for all $k \in \mathcal{S}$. It follows from (65) and Lemma EC.5 that the mapping $y \mapsto \hat{\alpha}_k(y)$ is nonincreasing on $(-\infty, \kappa]$ for all $k \in \mathcal{S}$. Therefore, $\alpha_{k_0}/m_{k_0} \le \hat{\alpha}_k(y)/m_k$ for all $k \in \mathcal{S}$ and $y \le \kappa$. By the definition of $\mathbf{A}$ in (EC.43), it then follows that $\mathbf{A}(y) = \alpha_{k_0}/m_{k_0}$ for all $y \le \kappa$, which implies that $\mathbf{A}$ is a constant function.

Case 2. Suppose that the assumption in the previous case does not hold, i.e., there exists $k_0 \in \{k \in \mathcal{S}_{\mathrm{w}} : \ell_k > 0\}$ such that $\hat{\alpha}_{k_0}(y)/m_{k_0} < \hat{\alpha}_k(y)/m_k$ for all $k \in \mathcal{S}^0$ in a neighborhood to the left of κ. It follows that $\mathbf{A}'(y) = -\ell_{k_0} < 0$ for all $y \in (y_1, \kappa)$, where y_1 is the largest point such that $\hat{\alpha}_{k_0}(y_1)/m_{k_0} = \hat{\alpha}_{k_1}(y_1)/m_{k_1}$ for some $k_1 \in \mathcal{S} \setminus \{k_0\}$.[10] But then $-\ell_{k_0} < -\ell_{k_1} \le 0$, $\lim_{y \nearrow y_1} \mathbf{A}'(y) = -\ell_{k_1}$, and $\lim_{y \searrow y_1} \mathbf{A}'(y) = -\ell_{k_0}$, implying that $\mathbf{A}$ is not differentiable at y_1. Continuing iteratively, we find the largest point y' such that there exists $k' \in \mathcal{S}^0$ with $\alpha_{k'}/m_{k'} \le \hat{\alpha}_k(y)/m_k$ for all $k \in \mathcal{S}$ and $y \le y'$. We conclude that $\mathbf{A}$ is constant up to y', after which it has a strictly negative derivative except at finitely many points where it is non-differentiable. The precise mathematical details are straightforward, though somewhat tedious, and are therefore omitted. □

Proof of Corollary EC.3. The product classes k_j are chosen iteratively as the minimizers at each breakpoint y_j, where y_j is defined as the largest point where the minimizer changes; see (EC.44)–(EC.45). Thus, on each interval $(y_{j+1}, y_j]$, the same product k_j attains the minimum. By Lemma EC.8, $\mathbf{A}$ has finitely many breakpoints. Therefore, the iterative construction of k_j and y_j terminates in a finite number of steps. Substituting this representation into (EC.37) and (EC.43) yields the desired result. □

Proof of Lemma EC.9. To show that v_γ^- is continuously differentiable on $[l_\gamma, w_1]$, we must establish both continuity and existence of a continuous derivative. First, continuity of v_γ^- on the interior of each subinterval in (EC.50) follows from the continuity of the functions $\hat{v}_{\gamma,k}^-$ for $k = 0, 1, \ldots, k^\star(l_\gamma)$ on their respective domains. Continuity of v_γ^- at the endpoints of these subintervals follows directly from the initial conditions (EC.52), (EC.56), and (EC.60). The existence of a continuous derivative follows from the continuity of the functions $\hat{v}_\gamma^-$ and $\phi(\cdot, 0)$, together with (EC.51)–(EC.52), (EC.55)–(EC.56), and (EC.59)–(EC.60). This establishes that $v_\gamma^- \in C^1[l_\gamma, w_1]$. Finally, it follows from the construction in (EC.51)–(EC.63)

[9] Since only products in $\mathcal{S}^0$ can achieve the minimum in (EC.43), there are at most $K - |\mathcal{S}^0| - \mathbf{1}_{\{|\mathcal{S}^0|=0\}}$ points where $\mathbf{A}$ is non-differentiable. However, this detail is not essential for our purposes.

[10] Since $\hat{\alpha}_{k_0}(y)/m_{k_0} \to \infty$ as $y \to -\infty$, there exists $\bar{k}_1 \in \mathcal{S}^0$ and $\bar{y}_1 < \kappa$ such that $\hat{\alpha}_{\bar{k}_1}(y)/m_{\bar{k}_1} < \hat{\alpha}_{k_0}(y)/m_{k_0}$ for $y < \bar{y}_1$, where $\hat{\alpha}_{\bar{k}_1}(\bar{y}_1)/m_{\bar{k}_1} = \hat{\alpha}_{k_0}(\bar{y}_1)/m_{k_0}$. This proves that y_1 exists.

that v_γ^- satisfies (EC.64)–(EC.65). The uniqueness of v_γ^- follows from the uniqueness of each function $\hat{v}_{\gamma,k}^-$ on its respective domain. □

Proof of Lemma EC.10. Fix $\gamma > 0$. Since $v_\gamma^-(l_\gamma) = 0 \le \kappa$ by (EC.52), it follows from (EC.49) and (EC.63) that $\phi_{k^\star(l_\gamma)}^-(l_\gamma) = \phi(l_\gamma, 0) = \gamma$. Substituting this into (EC.51)–(EC.52) yields $(v_\gamma^-)'(l_\gamma) = 0$. To complete the proof, we must show that $(v_\gamma^-)'(w) > 0$ for all $w \in (l_\gamma, w_1]$. By Lemma EC.7 and (EC.63), $(v_\gamma^-)'$ is almost everywhere differentiable on $[l_\gamma, w_1]$ with respect to Lebesgue measure. In particular, the second derivative of v_γ^- is given by

$$(v_\gamma^-)''(w) = \frac{m'H^{-1}m}{\sigma^2}\, v_\gamma^-(w)\,(v_\gamma^-)'(w) - \frac{2\mu}{\sigma^2}\,(v_\gamma^-)'(w) - \frac{2}{\sigma^2}\,\phi'(w,0), \;\; w \in [l_\gamma, w_1] \setminus \bigcup_{k=1}^{k^\star(l_\gamma)} \{\tau_{k^\star(l_\gamma)-k}^-\}, \tag{EC.102}$$

with $(v_\gamma^-)''(l_\gamma) = 2\,\hat{\beta}_{i^\star(k^\star(l_\gamma))} \,/\, (m_{i^\star(k^\star(l_\gamma))}\,\sigma^2) > 0$. From (EC.102), continuity of v_γ^- and $(v_\gamma^-)'$, and continuity of $\phi'(\cdot, 0)$ in a neighborhood of l_γ, we conclude that $(v_\gamma^-)''$ is continuous in a neighborhood of l_γ. Since $(v_\gamma^-)''(l_\gamma) > 0$, there exists $\bar{w} > l_\gamma$ such that $(v_\gamma^-)''(w) > 0$ for all $w \in [l_\gamma, \bar{w}]$. From this and $(v_\gamma^-)'(l_\gamma) = 0$, it follows that $(v_\gamma^-)'(w) > 0$ for all $w \in (l_\gamma, \bar{w}]$. It remains to show that $(v_\gamma^-)'(w) > 0$ for all $w \in (\bar{w}, w_1]$. Aiming for a contradiction, suppose that $(v_\gamma^-)'(w)$ is not strictly positive on $(\bar{w}, w_1]$. It follows that

$$\hat{w} := \inf\left\{w \in (\bar{w}, w_1] : (v_\gamma^-)'(w) = 0\right\}$$

is well-defined. From the definition of $\hat{w}$, we have that $(v_\gamma^-)'(\hat{w}) = 0$ and $(v_\gamma^-)'(w) > 0$ for all $w \in (l_\gamma, \hat{w})$. Since $v_\gamma^-(l_\gamma) = 0$ by (EC.52), we must have $v_\gamma^-(\hat{w}) > 0$. Substituting this into (EC.64) yields

$$\frac{\sigma^2}{2}\,(v_\gamma^-)'(\hat{w}) = \gamma - \phi(\hat{w}, 0) - \mu\, v_\gamma^-(\hat{w}) + \frac{m'H^{-1}m}{4}\,(v_\gamma^-)^2(\hat{w}) > 0,$$

where the inequality follows from $\gamma = \phi(l_\gamma, 0) > \phi(\hat{w}, 0)$ (which follows from Lemma EC.7 and $l_\gamma < \bar{w} \wedge w_0 \le \hat{w} \le w_1$) and the fact that the third and fourth terms in the equality are strictly positive (since $\mu < 0$ and $m'H^{-1}m{>}0$). This contradicts $(v_\gamma^-)'(\hat{w}) = 0$. Hence, the assumption that $(v_\gamma^-)'(w)$ is not strictly positive on $(\bar{w}, w_1]$ is incorrect, which completes the proof. □

Proof of Lemma EC.11. Fix γ_1 and γ_2 such that $0 \le \gamma_1 < \gamma_2$. Since $l_{\gamma_2} < l_{\gamma_1}$ and $v_{\gamma_2}^-$ is strictly increasing (by Lemma EC.10) with $v_{\gamma_2}^-(l_{\gamma_2}) = 0$ (by (EC.52)), it follows that $v_{\gamma_2}^-(l_{\gamma_1}) > 0 = v_{\gamma_1}^-(l_{\gamma_1})$. Aiming for a contradiction, suppose that there exists some $\bar{w} \in (l_{\gamma_1}, w_1]$ such that $v_{\gamma_1}^-(\bar{w}) \ge v_{\gamma_2}^-(\bar{w})$. It follows that

$$\hat{w} := \inf\left\{l_{\gamma_1} < w \le w_1 \,:\, v_{\gamma_1}^-(w) \ge v_{\gamma_2}^-(w)\right\}$$

is well-defined. By the intermediate value theorem, the continuity of $v_{\gamma_1}^-$ and $v_{\gamma_2}^-$, and the definition of $\hat{w}$, we have $v_{\gamma_1}^-(\hat{w}) = v_{\gamma_2}^-(\hat{w})$ and $v_{\gamma_2}^-(w) > v_{\gamma_1}^-(w)$ for all $w \in [l_{\gamma_1}, \hat{w})$. From (EC.64), it follows that

$$(v_{\gamma_2}^-)'(w) - (v_{\gamma_1}^-)'(w) > \frac{2}{\sigma^2}\,(\gamma_2 - \gamma_1) > 0 \quad \text{for all} \quad w \in [l_{\gamma_1}, \hat{w}).$$

However, this and $v_{\gamma_2}^-(l_{\gamma_1}) > v_{\gamma_1}^-(l_{\gamma_1})$ imply that $v_{\gamma_2}^-(\hat{w}) > v_{\gamma_1}^-(\hat{w})$, which is a contradiction. □

Proof of Lemma EC.12. By Lemma EC.11, the mapping $\gamma \mapsto v_\gamma^-(w_1)$ is strictly increasing. To show that $v_0^-(w_1) = 0$, note that $l_0 = w_0$, so it follows by (EC.62) that $v_0^- \in C^1[w_0, w_1]$ is the unique solution to the IVP from Step 3 of the construction, i.e.,

$$\begin{aligned} v'(w) &= \frac{m'H^{-1}m}{2\sigma^2}\, v^2(w) - \frac{2\mu}{\sigma^2}\, v(w), \quad w \in [w_0, \infty), \\ v(w_0) &= 0. \end{aligned}$$

Since the zero function satisfies this IVP, uniqueness implies that $v_0^-(w) = 0$ for all $w \in [w_0, w_1]$, and in particular, $v_0^-(w_1) = 0$, as desired. Finally, to prove that $\lim_{\gamma\to\infty} v_\gamma^-(w_1) = \infty$, fix an arbitrary $M > 0$. Then, choosing $\gamma = M\sigma^2/(2(w_1 - w_0)) > 0$, it follows that

$$\begin{aligned} v_\gamma^-(w_1) &= \int_{l_\gamma}^{w_1} (v_\gamma^-)'(w)\, dw \geq \int_{w_0}^{w_1} (v_\gamma^-)'(w)\, dw = \int_{w_0}^{w_1} (\hat{v}_{\gamma,0}^-)'(w)\, dw \\ &= \int_{w_0}^{w_1} \left(\frac{m'H^{-1}m}{2\sigma^2}\, (\hat{v}_{\gamma,0}^-)^2(w) - \frac{2\mu}{\sigma^2}\, \hat{v}_{\gamma,0}^-(w) + \frac{2\gamma}{\sigma^2} \right) dw \geq \frac{2\gamma}{\sigma^2}\, (w_1 - w_0) = M, \end{aligned}$$

where the first equality follows from $v_\gamma^-(l_\gamma) = 0$ by (EC.52), the first inequality follows from the fact that $(v_\gamma^-)'(w) > 0$ for all $w \in [l_\gamma, w_1]$ by Lemma EC.10, the second equality follows from (EC.62), the third equality follows from (EC.59), the second inequality follows from the fact that the first two terms in the third equality are nonnegative, and the final equality follows from the choice of γ. Since M was arbitrary, we conclude that $\lim_{n\to\infty} v_\gamma^-(w_1) = \infty$. □

Proof of Lemma EC.13. We show that for all $\gamma \geq 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $\gamma' \in (\gamma - \delta, \gamma + \delta) \cap [0, \infty)$, we have $|v_\gamma^-(x) - v_{\gamma'}^-(x)| < \epsilon$ for all $x \in [w_0, w_1]$. The desired result then follows by setting $x = w_1$. Fix $\gamma \geq 0$ and $\epsilon > 0$, and consider some $\delta \in (0, 1)$ to be specified later. For any $\gamma' \in (\gamma - \delta, \gamma + \delta) \cap [0, \infty)$ and $x \in [w_0, w_1]$, we have that

$$\begin{aligned} v_\gamma^-(x) - v_{\gamma'}^-(x) &= \int_{l_\gamma}^{x} (v_\gamma^-)'(w)\, dw - \int_{l_\gamma}^{x} (v_{\gamma'}^-)'(w)\, dw \\ &= \int_{l_\gamma}^{x} \left[\frac{m'H^{-1}m}{2\sigma^2}\, (v_\gamma^-)^2(w) - \frac{2\mu}{\sigma^2}\, v_\gamma^-(w) \right] dw - \int_{l_{\gamma'}}^{x} \left[\frac{m'H^{-1}m}{2\sigma^2}\, (v_{\gamma'}^-)^2(w) - \frac{2\mu}{\sigma^2}\, v_{\gamma'}^-(w) \right] dw \\ &\quad + \int_{l_\gamma}^{x} \frac{2}{\sigma^2}\, \phi(w, 0)\, dw - \int_{l_{\gamma'}}^{x} \frac{2}{\sigma^2}\, \phi(w, 0)\, dw + (x - l_\gamma)\, \frac{2\gamma}{\sigma^2} - (x - l_{\gamma'})\, \frac{2\gamma'}{\sigma^2}, \end{aligned}$$

where the first equality follows from the fundamental theorem of calculus and (EC.52), and the second equality from (EC.64) and rearranging terms. Taking absolute values of both sides of the previous display then gives

$$|v_\gamma^-(x) - v_{\gamma'}^-(x)| \leq \int_{l_\gamma \wedge l_{\gamma'}}^{l_\gamma \vee l_{\gamma'}} \left| \frac{m'H^{-1}m}{2\sigma^2}\, (v_{\gamma\vee\gamma'}^-)^2(w) - \frac{2\mu}{\sigma^2}\, v_{\gamma\vee\gamma'}^-(w) + \frac{2}{\sigma^2}\, \phi(w, 0) \right| dw$$

$$
\begin{aligned}
&+\int_{l_\gamma \vee l_{\gamma'}}^{x} \frac{m'H^{-1}m}{2\sigma^2}\left|(v_\gamma^-)^2(w)-(v_{\gamma'}^-)^2(w)\right|dw+\left|(x-l_\gamma)\frac{2\gamma}{\sigma^2}-(x-l_{\gamma'})\frac{2\gamma'}{\sigma^2}\right| \\
&\le \underbrace{\left|\frac{m'H^{-1}m}{2\sigma^2}(v_{\gamma+1}^-)^2(w_1)-\frac{2\mu}{\sigma^2}v_{\gamma+1}^-(w_1)+\frac{2}{\sigma^2}\phi(l_{\gamma+1},0)\right|}_{=:C_1(\gamma)}\cdot|l_\gamma-l_{\gamma'}| \\
&\quad+\underbrace{\frac{2}{\sigma^2}|w_1-l_{\gamma+1}|}_{=:C_2(\gamma)}\cdot|\gamma-\gamma'|+\int_{l_\gamma\vee l_{\gamma'}}^{x}\underbrace{\left(\frac{m'H^{-1}m}{\sigma^2}v_{\gamma+1}^-(w_1)-\frac{2\mu}{\sigma^2}\right)}_{=:C_3(\gamma)}|v_\gamma^-(w)-v_{\gamma'}(w)|\,dw \\
&=C_1(\gamma)\,|l_\gamma-l_{\gamma'}|+C_2(\gamma)\,|\gamma-\gamma'|+C_3(\gamma)\int_{l_\gamma\vee l_{\gamma'}}^{x}|v_\gamma^-(w)-v_{\gamma'}(w)|\,dw,
\end{aligned}
$$

where the first inequality follows from the triangle inequality and rearranging terms, and the second inequality from Lemmas EC.7, EC.10, and EC.11 and the inequality $l_{\gamma+1}\le l_\gamma\wedge l_{\gamma'}$.[11] Furthermore, by Lemma EC.7, the mapping $w\mapsto\phi(w,0)$ is continuous and injective, implying that $l_{\gamma'}\to l_\gamma$ as $\gamma'\to\gamma$.[12] Therefore, there exists some $\delta\in(0,1)$ such that for all $\gamma'\in(\gamma-\delta,\gamma+\delta)\cap[0,\infty)$ we have

$$
\widetilde{C}(\gamma,\gamma')\coloneqq C_1(\gamma)\,|l_\gamma-l_{\gamma'}|+C_2(\gamma)\,|\gamma-\gamma'|<\epsilon\exp\left\{-C_3(\gamma)\,(w-l_{\gamma+1})\right\}.
$$

Then, following the same argument as in the proof of Gronwall's inequality (see, e.g., Theorem 1.9.1 in Lakshmikantham and Leela (1969)), it follows from the previous two displayed equations that

$$
|v_\gamma^-(x)-v_{\gamma'}(x)|\le\widetilde{C}(\gamma,\gamma')\exp\left(\int_{l_\gamma\vee l_{\gamma'}}^{x}C_3(\gamma)\,dw\right)\le\widetilde{C}(\gamma,\gamma')\exp\left\{C_3(\gamma)\,(w_1-l_{\gamma+1})\right\}<\epsilon.
$$

for all $x\in[w_0,w_1]$. This completes the proof. □

Proof of Lemma EC.14. For $\gamma\in[0,\gamma_\kappa]$, it follows from Lemma EC.11 and (EC.66) that $v_\gamma^-(w)\le\kappa$ for all $w\in[l_\gamma,w_1]$. Thus, $\phi(w,v_\gamma^-(w))=\phi(w,0)$ for all $w\in[l_\gamma,w_1]$. From this and Lemma EC.9, it follows that v_γ^- is the unique solution to (EC.47)–(EC.48), completing the proof. □

Proof of Lemma EC.15. To show that v_γ^+ is continuously differentiable on $[w_1,u_\gamma]$, we must prove that it has a continuous derivative. For convenience, we decompose the interval $[w_1,u_\gamma]$ into two disjoint sets, $S_{\gamma,1}$ and $S_{\gamma,2}$, as follows:

$$
S_{\gamma,1}\coloneqq[w_1,\tau_{\gamma,B_\gamma}^+)\cup\bigcup_{j=1}^{B_\gamma-1}(\tau_{\gamma,j+1}^+,\tau_{\gamma,j}^+)\cup(\tau_{\gamma,1}^+,u_\gamma]\quad\text{and}\quad S_{\gamma,2}\coloneqq\bigcup_{j=1}^{B_\gamma}\{\tau_{\gamma,j}^+\}.
$$

[11] In particular, for the second inequality, since $0<\delta<1$, Lemmas EC.10 and EC.11 imply that $v_{\gamma+1}(w_1)\ge\max\{\sup\{v_\gamma^-(w):w\in[l_\gamma,w_1]\},\sup\{v_{\gamma'}^-(w):w\in[l_{\gamma'},w_1]\}\}$. Furthermore, by Lemma EC.7, $\phi(l_{\gamma+1},0)\ge\phi(w,0)$ for all $w\in[l_\gamma\wedge l_{\gamma'},w_1]$ since $l_{\gamma+1}\le l_\gamma\wedge l_{\gamma'}$.

[12] To be more precise, the function $\phi(\cdot,0)$ restricted to $(-\infty,w_0]$ is continuous and injective. Therefore, it has a continuous inverse $(\phi(\cdot,0)|_{(-\infty,w_0]})^{-1}:[0,\infty)\to(-\infty,w_0]$. Denoting this inverse as ϕ^{-1}, it follows that for any sequence $\{\gamma_n\}\subseteq[0,\infty)$ such that $\gamma_n\to\gamma$ as $n\to\infty$, we have $l_{\gamma_n}=\phi^{-1}(\gamma_n)\to\phi^{-1}(\gamma)=l_\gamma$ as $n\to\infty$, proving that $\gamma\mapsto l_\gamma$ is continuous, as desired.

We first prove that v_γ^+ is differentiable. Differentiability on $S_{\gamma,1}$ follows from the differentiability of the functions $\hat{v}_{\gamma,j}^+$ for $j = 0, 1, \ldots, B_\gamma$. Differentiability on $S_{\gamma,2}$ follows from

$$\hat{v}_{\gamma,j}^+(\tau_{\gamma,j+1}^+) = \hat{v}_{\gamma,j+1}^+(\tau_{\gamma,j+1}^+) \quad \text{and} \quad \phi_j^+(\tau_{\gamma,j+1}^+, y_{j+1}) = \phi_{j+1}^+(\tau_{\gamma,j+1}^+, y_{j+1}), \quad j = 0, 1, \ldots, B_\gamma - 1,$$

where the first equality follows from (EC.75) and (EC.79), and the second follows from (EC.45), (EC.72), (EC.76), and (EC.80). Next, we prove that the derivative of v_γ^+ is continuous. Continuity on $S_{\gamma,1}$ follows from the continuity of the functions $\hat{v}_{\gamma,j}^+$ for $j = 0, 1, \ldots, B_\gamma$ and (EC.70), (EC.72), (EC.74), (EC.76), (EC.78), and (EC.80). Continuity on $S_{\gamma,2}$ follows from the initial conditions (EC.75) and (EC.79). □

Proof of Lemma EC.16. Fix $\gamma \ge 0$. Since $\phi(u_\gamma, \kappa) = \phi_0^+(u_\gamma, \kappa)$ by Corollaries EC.2–EC.3 and (EC.72), it follows from (EC.69)–(EC.71) that $(v_\gamma^+)'(u_\gamma) = 0$. To complete the proof, we must show that $(v_\gamma^+)'(w) > 0$ for all $w \in [w_1, u_\gamma)$. As we will see shortly, it suffices to show that $(\hat{v}_{\gamma,0}^+)'(w) > 0$ for all $w \in (-\infty, u_\gamma)$ and that $(\hat{v}_{\gamma,j}^+)'(w) > 0$ for all $w \in (-\infty, \tau_{\gamma,j}^+]$ for $j = 1, \ldots, B_\gamma$. To that end, consider the function $\hat{v}_{\gamma,0}^+$. By (EC.70) and (EC.72), it follows that $\hat{v}_{\gamma,0}^+$ has a well-defined second derivative given by

$$(\hat{v}_{\gamma,0}^+)''(w) = \frac{m'H^{-1}m}{\sigma^2}\, \hat{v}_{\gamma,0}^+(w)\, (\hat{v}_{\gamma,0}^+)'(w) - \frac{2\mu}{\sigma^2}\, (\hat{v}_{\gamma,0}^+)'(w) - \frac{2}{\sigma^2}\frac{d}{dw}\, \phi_0^+\big(w, \hat{v}_{\gamma,0}^+(w)\big), \quad w \in (-\infty, u_\gamma],$$

where

$$\frac{d}{dw}\, \phi_0^+\big(w, \hat{v}_{\gamma,0}^+(w)\big) = \hat{\alpha}_{k_0}\big(\hat{v}_{\gamma,0}^+(w)\big)/m_{k_0} + (w - w_1)\, \ell_{k_0}(\hat{v}_{\gamma,0}^+)'(w), \quad w \in (-\infty, u_\gamma].$$

Since $\hat{v}_{\gamma,0}^+(u_\gamma) = \kappa$ and $(\hat{v}_{\gamma,0}^+)'(u_\gamma) = 0$ by (EC.69)–(EC.71), it follows from the above that $(\hat{v}_{\gamma,0}^+)''(u_\gamma) = -2\hat{\alpha}_{k_0}(\kappa)/(\sigma^2 m_{k_0}) < 0$. It follows that $\hat{v}_{\gamma,0}^+$ has a local maximum at u_γ, implying that there exists some $\bar{w}_0 < u_\gamma$ such that $(\hat{v}_{\gamma,0}^+(w))'(w) > 0$ for all $w \in [\bar{w}_0, u_\gamma)$. Now, aiming for a contradiction, suppose that $(\hat{v}_{\gamma,0}^+)'$ is not strictly positive on $(-\infty, \bar{w}_0)$. It follows that

$$\hat{w}_0 \coloneqq \sup\big\{w \in (-\infty, \bar{w}_0) : (\hat{v}_{\gamma,0}^+)'(w) = 0\big\}$$

is well-defined. Therefore, we have that $\hat{v}_{\gamma,0}^+(\hat{w}_0) < \kappa$, and by the continuity of $(\hat{v}_{\gamma,0}^+)'$, we also have that $(\hat{v}_{\gamma,0}^+)'(\hat{w}_0) = 0$. Similar to before, we have that $(\hat{v}_{\gamma,0}^+)''(\hat{w}_0) = -2\,\hat{\alpha}_{k_0}(\hat{v}_{\gamma,0}^+(\hat{w}_0))/(\sigma^2 m_{k_0}) < 0$. It follows that $\hat{v}_{\gamma,0}^+$ has a local maximum at $\hat{w}_0$, contradicting $(\hat{v}_{\gamma,0}^+)'(w) > 0$ for all $w \in (\hat{w}_0, u_\gamma)$. Thus, $(\hat{v}_{\gamma,0}^+)'(w) > 0$ for all $w \in (-\infty, u_\gamma)$. In particular, by (EC.82) and Lemma EC.15, it follows that $(v_\gamma^+)'(w) > 0$ for $w \in [\tau_{\gamma,1}^+, u_\gamma)$ and that $(\hat{v}_{\gamma,1}^+)'(\tau_{\gamma,1}^+) > 0$. The same argument can then be applied iteratively to show that $(\hat{v}_{\gamma,j}^+)'(w) > 0$ for all $w \in (-\infty, \tau_{\gamma,j}^+]$ for $j = 1, \ldots, B_\gamma$. Therefore, by Lemma EC.15, we conclude that $(v_\gamma^+)'(w) > 0$ for all $w \in [w_1, u_\gamma)$, completing the proof. □

Proof of Lemma EC.17. It follows from (EC.71) and (EC.82) that $v_\gamma^+(u_\gamma) = \hat{v}_{\gamma,0}^+(u_\gamma) = \kappa$. Therefore, v_γ^+ satisfies (EC.68). It remains to show that v_γ^+ satisfies (EC.67). To that end, by Corollary EC.3 and Lemma EC.16, we have that

$$k^\star(w, v_\gamma^+(w)) = \begin{cases} k_j^+, & w \in \left(\tau_{\gamma,j+1}^+, \tau_{\gamma,j}^+\right], \quad j = 0, 1, \ldots, B_\gamma - 1, \\ k_{B_\gamma}^+, & w \in \left[w_1, \tau_{\gamma,B_\gamma}^+\right]. \end{cases} \tag{EC.103}$$

It then follows from Corollary EC.2, together with (EC.72), (EC.76), (EC.80), and (EC.103), that

$$\phi(w, v_\gamma^+(w)) = \begin{cases} \phi_j^+(w, v_\gamma^+(w)), & w \in \left(\tau_{\gamma,j+1}^+, \tau_{\gamma,j}^+\right], \quad j = 0, 1, \ldots, B_\gamma - 1, \\ \phi_{B_\gamma}^+(w, v_\gamma^+(w)), & w \in \left[w_1, \tau_{\gamma,B_\gamma}^+\right]. \end{cases} \tag{EC.104}$$

Therefore, by (EC.70), (EC.74), (EC.78), and (EC.104), we conclude that v_γ^+ satisfies (EC.67). □

Proof of Lemma EC.18. Fix γ_1 and γ_2 such that $0 \le \gamma_1 < \gamma_2$. Since $u_{\gamma_2} > u_{\gamma_1}$ and $v_{\gamma_2}^+$ is strictly increasing (by Lemma EC.16) with $v_{\gamma_2}^+(u_{\gamma_2}) = \kappa$ (by (EC.71)), it follows that $v_{\gamma_2}^+(u_{\gamma_1}) < \kappa = v_{\gamma_1}^+(u_{\gamma_1})$. Aiming for a contradiction, suppose that there exists some $\bar{w} \in [w_1, u_{\gamma_1})$ such that $v_{\gamma_2}^+(\bar{w}) \ge v_{\gamma_1}^+(\bar{w})$. It follows that

$$\hat{w} \coloneqq \sup\left\{w_1 \le w < u_{\gamma_1} : v_{\gamma_2}^+(w) \ge v_{\gamma_1}^+(w)\right\}$$

is well-defined. By the intermediate value theorem, the continuity of $v_{\gamma_1}^+$ and $v_{\gamma_2}^+$, and the definition of $\hat{w}$, we have $v_{\gamma_1}^+(\hat{w}) = v_{\gamma_2}^+(\hat{w})$ and $v_{\gamma_1}^+(w) > v_{\gamma_2}^+(w)$ for all $w \in (\hat{w}, u_{\gamma_1}]$. From (EC.67), it follows that

$$(v_{\gamma_1}^+)'(\hat{w}) - (v_{\gamma_2}^+)'(\hat{w}) = \frac{2}{\sigma^2}\,(\gamma_1 - \gamma_2) < 0.$$

To complete the proof, observe that for each positive integer n such that $\hat{w} + n^{-1} < u_{\gamma_1}$, we have

$$n\left[\left(v_{\gamma_1}^+(\hat{w}+n^{-1}) - v_{\gamma_2}^+(\hat{w}+n^{-1})\right) - \left(v_{\gamma_1}^+(\hat{w}) - v_{\gamma_2}^+(\hat{w})\right)\right] = n\left(v_{\gamma_1}^+(\hat{w}+n^{-1}) - v_{\gamma_2}^+(\hat{w}+n^{-1})\right) > 0.$$

By the mean value theorem, for each such integer n, there exists $c_n \in (\hat{w}, \hat{w}+n^{-1})$ such that $(v_{\gamma_1}^+)'(c_n) - (v_{\gamma_2}^+)'(c_n) > 0$. Since $\lim_{n\to\infty} c_n = \hat{w}$, it follows from the continuity of $(v_{\gamma_1}^+)'$ and $(v_{\gamma_2}^+)'$ that $(v_{\gamma_1}^+)'(\hat{w}) - (v_{\gamma_2}^+)'(\hat{w}) = \lim_{n\to\infty}\left[(v_{\gamma_1}^+)'(c_n) - (v_{\gamma_2}^+)'(c_n)\right] \ge 0$. However, this contradicts $(v_{\gamma_2}^-)'(\hat{w}) > (v_{\gamma_1}^-)'(\hat{w})$. □

Proof of Lemma EC.19. By Lemma EC.16, it follows that the mapping $\gamma \mapsto v_\gamma^+(w_1)$ is strictly decreasing. To complete the proof, we must show that $v_0^+(w_1) \in (0, \kappa)$. First, by Lemma EC.7 and (EC.69), we have that $w_1 < u_0$. Moreover, since v_0^+ is strictly increasing on $[w_1, u_0]$ (by Lemma EC.16) with $v_0^+(u_0) = \kappa$ (by (EC.68)), it follows that $v_0^+(w_1) < \kappa$. To show that $v_0^+(w_1) > 0$, assume for contradiction that $v_0^+(w_1) \le 0$. It then follows from the continuity of v_0^+ and the intermediate value theorem that there exists some $\bar{w} \in [w_1, u_0)$ such that $v_0^+(\bar{w}) = 0$. From (EC.67), this implies that

$$(v_0^+)'(\bar{w}) = -\frac{2}{\sigma^2}\,\phi(\bar{w}, 0) \le 0,$$

where the inequality follows from Lemma EC.7 and since $\bar{w} \ge w_1$. However, this contradicts Lemma EC.16. We conclude that $v_0^+(w_1) > 0$, which completes the proof. □

Proof of Lemma EC.20. Similar to the proof of Lemma EC.13, we must show that for all $\gamma \geq 0$ and $\epsilon > 0$, there exists $\delta = \delta(\gamma, \epsilon) > 0$ such that for all $\gamma' \in (\gamma - \delta, \gamma + \delta) \cap [0, \infty)$, we have $|v^+_\gamma(w_1) - v^+_{\gamma'}(w_1)| < \epsilon$. The added complexity here is that both the number of breakpoints B_γ and their values $\tau^+_{\gamma,j}$ for $j = 0, 1, \ldots, B_\gamma$ are functions of γ. However, it can be shown that these breakpoints also vary continuously with γ. Thus, the mathematical argument closely mirrors that of Lemma EC.13 and is therefore omitted. □

Proof of Lemma EC.21. First, we show that there exists a unique $\gamma^\star \in [0, \gamma_\kappa]$ such that $v^-_{\gamma^\star}(w_1) = v^+_{\gamma^\star}(w_1)$. Define the function $\tilde{v} : [0, \gamma_\kappa] \to \mathbb{R}$ as follows:

$$\tilde{v}(\gamma) := v^+_\gamma(w_1) - v^-_\gamma(w_1), \quad \gamma \in [0, \gamma_\kappa].$$

By Lemmas EC.12 and EC.19, it follows that $\tilde{v}$ is strictly decreasing on $[0, \gamma_\kappa]$, with

$$\tilde{v}(0) = v^+_0(w_1) \in (0, \kappa) \quad \text{and} \quad \tilde{v}(\gamma_\kappa) = v^+_{\gamma_\kappa}(w_1) - v^-_{\gamma_\kappa}(w_1) < 0. \tag{EC.105}$$

Moreover, Lemmas EC.13 and EC.20 imply $\tilde{v}$ is continuous on $[0, \gamma_\kappa]$. Thus, by the intermediate value theorem, the strictly decreasing nature of $\tilde{v}$, and (EC.105), there exists a unique $\gamma^\star \in [0, \gamma_\kappa]$ such that $\tilde{v}(\gamma^\star) = 0$. In particular, noting that $\tilde{v}(0) > 0$ and $\tilde{v}(\gamma^\star) = 0$, it follows immediately that $\gamma^\star > 0$.

To complete the proof, it remains to show that $v_{\gamma^\star} \in C^1[l_{\gamma^\star}, u_{\gamma^\star}]$. By Lemmas EC.9 and EC.15, together with (EC.83), we know that $v_{\gamma^\star}$ is continuously differentiable on $[l_{\gamma^\star}, w_1) \cup (w_1, u_{\gamma^\star}]$. Since we have already established that $v^-_{\gamma^\star}(w_1) = v^+_{\gamma^\star}(w_1)$, continuity at w_1 follows immediately. Therefore, it remains to show that the derivative $v'_{\gamma^\star}$ exists and is continuous at w_1. Since $v_{\gamma^\star}$ is continuous at w_1 and differentiable both to the left and right of w_1, it suffices to show that $\lim_{w \nearrow w_1} v'_{\gamma^\star}(w) = \lim_{w \searrow w_1} v'_{\gamma^\star}(w)$. However, by (EC.47) and (EC.67), together with the continuity of $v_{\gamma^\star}$, this reduces to verifying that $\lim_{w \nearrow w_1} \phi(w, v^-_{\gamma^\star}(w)) = \lim_{w \searrow w_1} \phi(w, v^+_{\gamma^\star}(w))$. To that end, note that

$$\lim_{w \nearrow w_1} \phi\big(w, v^-_{\gamma^\star}(w)\big) = 0 = \lim_{w \searrow w_1} \frac{\hat{\alpha}_{k^\star(w, v^+_{\gamma^\star}(w))}\big(v^+_{\gamma^\star}(w)\big)\,(w - w_1)}{m_{k^\star(w, v^+_{\gamma^\star}(w))}} = \lim_{w \searrow w_1} \phi\big(w, v^+_{\gamma^\star}(w)\big),$$

where the first equality follows from Corollary EC.2 for $(w, y) \in [w_0, w_1] \times (-\infty, \kappa]$ (since $v^-_{\gamma^\star}(w) \leq \kappa$ for all $w \in [l_{\gamma^\star}, w_1]$), the second equality since $\hat{\alpha}_{k^\star(w, v^+_{\gamma^\star}(w))}(v^+_{\gamma^\star}(w)) / m_{k^\star(w, v^+_{\gamma^\star}(w))}$ is bounded on $[w_1, u_\gamma]$ (since $v^+_{\gamma^\star}(w)$ is bounded on $[w_1, u_\gamma]$), and the third equality from Corollary EC.2 for $(w, y) \in [w_1, \infty) \times (-\infty, \kappa]$ (since $v^+_{\gamma^\star}(w) \leq \kappa$ for all $w \in [w_1, u_{\gamma^\star}]$). We conclude that $v'_{\gamma^\star}$ exists and is continuous at w_1. □

Proof of Corollary EC.4. From (EC.47)–(EC.49), it follows that $v_{\gamma^\star}(l_{\gamma^\star}) = v^-_{\gamma^\star}(l_{\gamma^\star}) = 0$ and $v'_{\gamma^\star}(l_{\gamma^\star}) = (v^-_{\gamma^\star})'(l_{\gamma^\star}) = 0$. Similarly, from (EC.67)–(EC.69), it follows that $v_{\gamma^\star}(u_{\gamma^\star}) = v^+_{\gamma^\star}(u_{\gamma^\star}) = \kappa$ and $v'_{\gamma^\star}(u_{\gamma^\star}) = (v^+_{\gamma^\star})'(u_{\gamma^\star}) = 0$. Therefore, $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ satisfies (60). Next, from (EC.47) and Lemma EC.14, it follows that $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v^-_{\gamma^\star})$ satisfies (59) for $w \in [l_{\gamma^\star}, w_1]$. Similarly, from (EC.67) and Lemma EC.17, it follows that $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v^+_{\gamma^\star})$ satisfies (59) for $w \in [w_1, u_{\gamma^\star}]$. Therefore, $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ satisfies (59) for all $w \in [l_{\gamma^\star}, u_{\gamma^\star}]$. We conclude that the tuple $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ is a solution to the Bellman equation (59)–(60).

Since $\gamma^\star > 0$ by Lemma EC.21, it follows from (EC.49), (EC.69), and Lemma EC.7 that $l_{\gamma^\star} < w_0$ and $u_{\gamma^\star} > w_1$. Finally, by Lemmas EC.10, EC.16, and EC.21, the function $v_{\gamma^\star}$ is continuous, with $v^-_{\gamma^\star}$ strictly increasing on $[l_{\gamma^\star}, w_1]$ and $v^+_{\gamma^\star}$ strictly increasing on $[w_1, u_{\gamma^\star}]$. It follows from (EC.83) that $v_{\gamma^\star}$ is strictly increasing on $[l_{\gamma^\star}, u_{\gamma^\star}]$. In particular, since $v_{\gamma^\star}(l_{\gamma^\star}) = 0$, we conclude that $v_{\gamma^\star}$ is nonnegative. □

Proof of Lemma EC.22. Let (l, u, γ, v) be a solution to the Bellman equation (59)–(60). Then, by definition, we have that $l < u$. We will establish that $l < w_0$, $u > w_1$, and $0 \le \gamma \le \gamma_\kappa$ via contradiction. First, suppose $\gamma < 0$. Then it follows from (59)–(60) that $\phi(l, 0) = \gamma < 0$, contradicting Lemma EC.7. Thus, we conclude that $\gamma \ge 0$.

Second, suppose $u \le w_1$. We consider the following two cases: $w_0 \le u \le w_1$ and $u < w_0$. If $w_0 \le u \le w_1$, it follows from Lemma EC.7 that $\phi(u, \kappa) = 0$. However, from (59)–(60), we also have $\phi(u, \kappa) = \gamma - \mu\kappa + m' H^{-1} m \kappa^2/4 > 0$, which is a contradiction. If instead $u < w_0$, it follows from Lemma EC.7 that $u < l$ (since $\gamma < \gamma - \mu\kappa + m' H^{-1} m \kappa^2/4$). However, this contradicts the fact that $l < u$. Since both cases yield contradictions, we conclude that $u > w_1$. Having established that $u > w_1$, note that since $w \mapsto \phi(w, \kappa)$ is strictly increasing on $[w_1, \infty)$, and hence injective, it follows that $u = u_\gamma$, where u_γ is given by (EC.69).

Third, suppose $l \ge w_0$. Since $u = u_\gamma$, it follows from (59)–(60) that v satisfies (EC.67)–(EC.68). Therefore, by the uniqueness of the solution $v^+_\gamma \in C^1[w_1, u_\gamma]$ to (EC.67)–(EC.68), it follows that $v(w) = v^+_\gamma(w)$ for $w \in [l \vee w_1, u_\gamma]$. In particular, by Lemma EC.16, we have that $v'(l \vee w_1) > 0$. Now consider the following two cases: $w_0 \le l \le w_1$ and $l > w_1$. If $w_0 \le l \le w_1$, then it follows from (60) and Corollary EC.2 that $\gamma = 0$. Substituting $\gamma = 0$ into (59)–(60), we see that v satisfies (EC.47)–(EC.48) with γ and l_γ replaced by 0 and l, respectively. Since the zero function satisfies this IVP, it follows from the uniqueness of the construction in Appendix EC.2.2.3.1 that $v(w) = 0$ for $w \in [l, w_1]$. But, since v is continuously differentiable over $[l, u]$, we have that $v'(w_1) = 0$. However, this contradicts $v'(l \vee w_1) = v'(w_1) > 0$. If instead $l > w_1$, it follows that $v'(l \vee w_1) = v'(l) > 0$, contradicting (60). Since both cases yield contradictions, we conclude that $l < w_0$. Having established that $l < w_0$, note that since $w \mapsto \phi(w, 0)$ is strictly decreasing on $(-\infty, w_0]$, and hence injective, it follows that $l = l_\gamma$, where l_γ is given by (EC.49).

Finally, we show that $\gamma \le \gamma_\kappa$. It suffices to show that $v(w) \le \kappa$ for all $w \in [l, u]$. (If $v(w) \le \kappa$ for all $w \in [l, u]$ but $\gamma > \gamma_\kappa$, then from (59)–(60), (EC.47)–(EC.48), (EC.66), and Lemma EC.12, it would follow that $v(w_1) > \kappa$, which is a contradiction.) Aiming for a contradiction, suppose that $v(w) > \kappa$ for some $w \in [l, u]$. Since $v \in C^1[l, u]$ and satisfies (60), there must exist some $\hat{w} \in (l, u)$ such that $v(\hat{w}) > \kappa$ and $v'(\hat{w}) < 0$. To arrive at a contradiction, it suffices to show that $v'(\hat{w}) > v'(u)$ (since $v'(u) = 0$ by (60), this would contradict $v'(\hat{w}) < 0$). From (59), together with $v(\hat{w}) > \kappa$ and $v(u) = \kappa$, it is enough to show that

$\phi(\hat{w}, v(\hat{w})) \leq \phi(u,\kappa)$. Since $v(\hat{w}) > \kappa$, it follows from (65) that $\hat{\alpha}_k(v(\hat{w})) \leq \hat{\alpha}_k(y)$ for all $y \in (-\infty,\kappa]$ and $k \in \mathcal{S}$. By (58) and (66)–(67), we then have that $\phi(\hat{w}, v(\hat{w})) \leq \phi(\hat{w}, y)$ for all $y \in (-\infty,\kappa]$. Therefore, to complete the proof, it is enough to show that $\phi(\hat{w},\kappa) \leq \phi(u,\kappa)$. To do so, we consider the following two cases: $w_0 \leq \hat{w} < u$ and $l < \hat{w} < w_0$. If $w_0 \leq \hat{w} < u$, then the desired inequality follows immediately from Lemma EC.7. If instead $l < \hat{w} < w_0$, then it follows from (59)–(60) that $\phi(l,0) = \gamma < \phi(u,\kappa)$. Furthermore, by (EC.38) and Corollary EC.2, we have that $\phi(l,0) = \phi(l,\kappa)$. It then follows from Lemma EC.7 that $\phi(\hat{w},\kappa) < \phi(l,\kappa) = \gamma < \phi(u,\kappa)$, as desired. This completes the proof. □

Proof of Corollary EC.5. From Lemma EC.22, any solution (l,u,γ,v) to the Bellman equation (59)–(60) must satisfy $l = l_\gamma$, $u = u_\gamma$, and $\gamma \in [0,\gamma_\kappa]$. But then, by (59)–(60), v must also satisfy the IVPs (EC.47)–(EC.48) and (EC.67)–(EC.68). By Lemmas EC.14 and EC.17, uniqueness implies that $v = v_\gamma^-$ on $[l, w_1]$ and $v = v_\gamma^+$ on $(w_1, u]$. Therefore, by (EC.83), it follows that $v = v_\gamma$. Moreover, since v is continuously differentiable, it follows that $\gamma = \gamma^\star$, where $\gamma^\star$ is given by Lemma EC.21 (otherwise v would not be continuous at w_1). We conclude that the tuple $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ from Corollary EC.4 is the unique solution to (59)–(60). □

EC.5. Supplementary Material for the Numerical Study

EC.5.1. Semi-Markov Decision Process Formulation

This section describes a semi-Markov decision process (SMDP) formulation of the joint dynamic pricing, scheduling, and rejection control problem introduced in Section 3; for a detailed review of the theory of Markov decision processes, see Bertsekas (2012) and Puterman (2014). In this formulation, we assume that production times are exponentially distributed and replace the sojourn times $w_k(t)$ in the objective function (13) by $Q_k(t)/\lambda_k^\star$ for $k \in \mathcal{S}$ and $t \geq 0$. This formulation is used in Section 8 as a benchmark to evaluate the performance of our proposed control policy from Section 7. Under this SMDP model, the control decisions, namely, the pricing, scheduling, and rejection decisions, are updated only when an order arrives, a product is produced, or a customer abandons. We refer to these moments in time as decision epochs. Due to the Markovian nature of the system, the time between consecutive decision epochs is exponentially distributed. We now describe the main components of the SMDP formulation: the state space, action space, transition probabilities, and rewards.

State Space. We denote the system state by $q(t) = (q_k(t))$ for $t \geq 0$, where $q_k(t)$ denotes the number of outstanding orders of product k at time t. The state space, denoted by $\mathcal{Q}$, is as follows:

$$\mathcal{Q} := \left\{(q_k) : q_k \in \mathbb{Z} \text{ for } k \in \mathcal{S},\ 0 \leq q_k \leq M_k \text{ for } k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{o}}^{\text{MTS}}, -M_k \leq q_k \leq M_k \text{ for } k \in \mathcal{S}_{\text{w}}^{\text{MTS}}\right\}.$$

Observe that for computational feasibility, we truncate the queue length of product k for $k \in \mathcal{S}$ at $\pm M_k$, where $M_k \in \mathbb{N}$ is a tuning parameter. As detailed in Appendix EC.5.2, we use policy iteration to numerically solve the SMDP. Due to the high memory requirement of the policy evaluation step, we can only solve the SMDP with up to about 100 million states. In the four-product example discussed in Section 8, we use $M_k = 75$ for $k \in \mathcal{S}$. This appears to contain the likely states; see Figure EC.1.

Actions. At each decision epoch, the system manager makes a decision consisting of four components (p, s, u, o), where p is the pricing decision, s is the production decision, u is the reallocation decision, and o is the order rejection decision.

- Pricing Decisions: The system manager chooses a price vector $p = (p_k)$, where p_k denotes the price of product $k \in \mathcal{S}$. The set of admissible price vectors is denoted by $\bar{\mathcal{P}} = \mathbb{R}_+^K$.
- Production Decisions: The system manager chooses which product to produce, if any. The production decision is denoted by $s \in \{0\} \cup \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$, where $s = 0$ denotes the decision to idle the server, and $s = k$ denotes the decision to produce product $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$.
- Reallocation Decisions: The system manager chooses whether to reallocate (or transfer) finished goods inventory from walk-in MTS class $w(k)$ to online MTS class k. The reallocation decision for product $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$ is denoted by $u_k \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, where $u_k = l$ denotes the decision to reallocate l units of finished goods inventory from class $w(k) \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ to class k.
- Order Rejection Decisions: The system manager chooses whether to reject (i.e., deny entry to) the next order of product k. The rejection decision for product $k \in \mathcal{S}$ is denoted by $o_k \in \{0, 1\}$, where $o_k = 1$ indicates rejecting the order and $o_k = 0$ indicates accepting the order.

Given the description of the actions above, the action space $\mathcal{A}$ is given as follows:

$$\mathcal{A} := \left\{(p, s, u, o) : p \in \bar{\mathcal{P}},\ s \in \{0\} \cup \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}},\ o \in \{0, 1\}^K,\ u_k \in \mathbb{Z}_+ \text{ for } k \in \mathcal{S}_{\text{o}}^{\text{MTS}}\right\}.$$

Policies. A policy specifies the decision rule to be used at each decision epoch. To describe the class of policies for the SMDP formulation, we first define the set of feasible actions at each state. Denote by $\mathcal{A}(q)$ the set of feasible actions at state $q \in \mathcal{Q}$, given as follows:

$$\begin{aligned}\mathcal{A}(q) := \Big\{(p, s, u, o) \in \mathcal{A} :\ & s \neq k \text{ if } q_k = 0 \text{ for } k \in \mathcal{S}^{\text{MTO}},\ s \neq k \text{ if } q_k = -M_k \text{ for } k \in \mathcal{S}_{\text{w}}^{\text{MTS}}, \\ & u_k \leq q_k \text{ for } k \in \mathcal{S}_{\text{o}}^{\text{MTS}},\ \textstyle\sum_{j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k)} u_j \leq q_k^- \text{ for } k \in \mathcal{S}_{\text{w}}^{\text{MTS}},\ o_k = 1 \text{ if } q_k = M_k \text{ for } k \in \mathcal{S}\Big\}.\end{aligned}$$

The requirement that $o_k = 1$ if $q_k = M_k$ for $k \in \mathcal{S}$ ensures that incoming orders for product k are rejected when the queue length of product k is equal to its maximum allowable value M_k. The requirements that $s \neq k$ if $q_k = 0$ for $k \in \mathcal{S}^{\text{MTO}}$ and $s \neq k$ if $q_k = -M_k$ for $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ ensure that production does not occur when the queue length of product k is equal to its minimum allowable value. Finally, the requirements that $u_k \leq q_k$ for $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$ and $\sum_{j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k)} u_j \leq q_k^-$ for $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ ensure that the system manager does not reallocate more finished goods inventory to the online classes than necessary, and does not reallocate more from walk-in MTS class $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ than its available finished goods inventory, respectively.

A policy is a mapping $\pi : \mathcal{Q} \to \mathcal{A}$ such that $\pi(q) \in \mathcal{A}(q)$ for $q \in \mathcal{Q}$. That is, we restrict attention to stationary Markov policies. The system manager's problem is to choose a policy π that maximizes the long-run average profit over an infinite horizon. For ease of notation, given an action $a \in \mathcal{A}$, we let $p_k(a)$, $\Lambda_k(a)$, $s(a)$, $u_k(a)$, and $o_k(a)$ denote the pricing decision, demand rate, production decision, reallocation decision, and order rejection decision for product k, respectively.

Uniformization. As is standard in the dynamic programming literature, we apply the uniformization technique to write the Bellman equation. To that end, let

$$\Psi := \sup_{p \in \bar{\mathcal{P}}} \sum_{k \in \mathcal{S}} \Lambda_k(p) + \max\{\mu_k : k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}\} + \sum_{k \in \mathcal{S}_{\text{w}}} \ell_k M_k < \infty,$$

which serves as an upper bound on the transition rates. In the uniformized SMDP, the time between two consecutive events is exponentially distributed with rate Ψ. After each event, an action $a \in \mathcal{A}(q)$ is chosen, where $q \in \mathcal{Q}$ denotes the state immediately following the event. The action is not updated until the subsequent event. Under action $a \in \mathcal{A}(q)$, the next event corresponds to the arrival of an order for product $k \in \mathcal{S}$ with probability $\Lambda_k(a)/\Psi$, the production of product $s(a)$ (provided $s(a) \neq 0$) with probability $\mu_{s(a)}/\Psi$, the abandonment of a customer from class $k \in \mathcal{S}_{\text{w}}$ with probability $\ell_k q_k^+/\Psi$, and a fictitious transition otherwise, in which case the system state remains unchanged.

Transition Probabilities. In the uniformized SMDP, the probability of transitioning from state $q \in \mathcal{Q}$ to state $q' \in \mathcal{Q}$ at the next event under action $a \in \mathcal{A}(q)$ is given as follows:

$$P_{qq'}(a) := \begin{cases} \Lambda_k(a)\mathbf{1}_{\{o_k(a)=0\}}/\Psi, & \text{if } q' = q^a + e_k,\ k \in \mathcal{S}, \\ \sum_{k \in \mathcal{S}} \Lambda_k(a)\mathbf{1}_{\{o_k(a)=1\}}/\Psi, & \text{if } q' = q^a,\ q^a \neq q, \\ (\mu_k \mathbf{1}_{\{s(a)=k\}} + \ell_k q_k^+)/\Psi, & \text{if } q' = q^a - e_k,\ k \in \mathcal{S}_{\text{w}}, \\ \mu_k \mathbf{1}_{\{s(a)=k\}}/\Psi, & \text{if } q' = q^a - e_k,\ k \in \mathcal{S}_{\text{o}}^{\text{MTO}}, \\ 1 - \sum_{\tilde{q} \neq q} P_{q\tilde{q}}(a), & \text{if } q' = q, \\ 0, & \text{otherwise}, \end{cases} \tag{EC.106}$$

where $e_k \in \mathbb{R}^K$ for $k \in \mathcal{S}$ denotes the standard unit basis vector whose kth component equals one and all other components equal zero, $q^a := q + \sum_{k \in \mathcal{S}_{\text{o}}^{\text{MTS}}} u_k(a)(e_{w(k)} - e_k)$ denotes the intermediate state in which reallocation decisions are implemented, and $\mu_0 := 0$. Note that the reallocation decisions are taken and implemented (i.e., products are reallocated) at the decision epochs—namely, when a customer order arrives, a product is produced, or a walk-in customer abandons. Although the system state may change after a reallocation decision is implemented, the system manager is not permitted to take another action (e.g., reallocate additional products) until the next decision epoch.

Rewards. The (one-stage) expected reward, i.e., the expected reward earned until the next event, at state $q \in \mathcal{Q}$ under action $a \in \mathcal{A}(q)$ is given by

$$R(q,a) := \sum_{k \in \mathcal{S}} \frac{\Lambda_k(a)}{\Psi} \Big(p_k(a) - (\gamma_k + r_k)\mathbf{1}_{\{o_k(a)=1\}}\Big) - \sum_{k \in \mathcal{S}} \frac{\Lambda_k(a)}{\Psi} \upsilon_k(q_k^a/\lambda_k^\star - \delta_k)\mathbf{1}_{\{o_k(a)=0\}}$$

$$- \frac{\mu_{s(a)}}{\Psi} \gamma_{s(a)} \mathbf{1}_{\{s(a)\neq 0\}} - \sum_{k\in\mathcal{S}_\mathrm{w}} \frac{\ell_k [q_k^a]^+}{\Psi} (\gamma_k + d_k) - \sum_{k\in\mathcal{S}_\mathrm{w}^\mathrm{MTS}} \frac{[q_k^a]^-}{\Psi} h_k. \tag{EC.107}$$

Here, the first term denotes the expected revenue, the second term serves as a surrogate for the earliness and tardiness costs, the third term captures the expected production cost, the fourth term accounts for the expected abandonment cost, and the last term represents the expected holding cost.

Bellman Equation. Next, we introduce the Bellman equation for long-run average cost dynamic programming problems, which characterizes an optimal policy:

$$\gamma + f(q) = \sup_{a\in\mathcal{A}(q)} \big\{ R(q,a) + \sum_{q'\in Q} P_{qq'}(a) f(q') \big\}, \quad q\in Q.$$

Here, $\gamma \in \mathbb{R}$ is often interpreted as a guess at the optimal long-run average expected profit. The function $f : Q \to \mathbb{R}$ is often called a relative value function in average cost dynamic programming problems. It is easy to see that the relative value function can only be determined up to an additive constant, even if γ is treated as a known constant. Therefore, we set $f(e_0) = 0$, where $e_0 = (0,\dots,0)$.

EC.5.2. Solving the Semi-Markov Decision Process Formulation

This section describes the policy iteration algorithm used to solve the Bellman equation for the SMDP formulation. To initialize the algorithm, we set $\gamma_0 \coloneqq 0$ and consider an initial policy π_0 that makes the following pricing, rejection, production, and reallocation decisions. First, it uses the price vector $\Lambda^{-1}(\lambda^\star)$ for all products. Second, it accepts all product $k\in\mathcal{S}$ orders except when $q_k = M_k$, in which case the order is rejected. Third, it produces the product $k\in\mathcal{S}^\mathrm{MTO}\cup\mathcal{S}_\mathrm{w}^\mathrm{MTS}$ with the largest backlog among the products that have a backlog; if no product has a backlog, no production occurs. Finally, for each $k\in\mathcal{S}_\mathrm{o}^\mathrm{MTS}$, it reallocates $[q_k - \lambda_k^\star \delta_k]^+$ units of class $w(k)$ inventory to class k. If the available walk-in MTS inventory is insufficient to satisfy all reallocation decisions, the system manager prioritizes the online MTS products in ascending order of their product class index. Starting with the initial policy π_0, we iteratively derive policies π_i for $i\in\mathbb{N}$ as follows:

Policy Evaluation. Given the policy $\pi_{i-1} : Q \to \mathcal{A}$, we solve for the function $f_{i-1} : Q \to \mathbb{R}$ and the scalar $\gamma_{i-1}\in\mathbb{R}$ that satisfy

$$\gamma_{i-1} + f_{i-1}(q) = R(q,\pi_{i-1}(q)) + \sum_{q'\in Q} P_{qq'}(\pi_{i-1}(q))\, f_{i-1}(q'), \quad q\in Q, \tag{EC.108}$$

along with the condition $f_{i-1}(e_0) = 0$. Equation (EC.108) corresponds to a sparse system of linear equations, which we solve using the BiCGSTAB iterative sparse solver from the C++ library Eigen.

Policy Improvement. Given the function $f_{i-1} : Q \to \mathbb{R}$, we define the updated policy $\pi_i : Q \to \mathcal{A}$ as follows:

$$\pi_i(q) \coloneqq \operatorname*{arg\,max}_{a\in\mathcal{A}(q)} \big\{ R(q,a) + \sum_{q'\in Q} P_{qq'}(a) f_{i-1}(q') \big\}, \quad q\in Q. \tag{EC.109}$$

Termination. The algorithm terminates when the actions under the policy π_i are sufficiently close to those under the previous policy π_{i-1}. To be specific, given an error threshold $\epsilon > 0$, we repeat the policy evaluation and policy improvement steps until, for some $i \in \mathbb{Z}_+$, we have

$$\left|p_k(\pi_i(q)) - p_k(\pi_{i-1}(q))\right| \le \epsilon,\ \ s(\pi_i(q)) = s(\pi_{i-1}(q)),\ \ u_k(\pi_i(q)) = u_k(\pi_{i-1}(q)),\ \ o_k(\pi_i(q)) = o_k(\pi_{i-1}(q))$$

for all $q \in \mathcal{Q}$ and $k \in \mathcal{S}$.

To facilitate the computation of the updated policy π_i, we show that, given the reallocation decisions, the updated pricing, rejection, and production decisions can be easily characterized. To simplify the analysis, we first write out the definition of the updated policy π_i. Specifically, substituting (EC.106)–(EC.107) into (EC.109) and removing terms independent of the actions yields the following:

$$\begin{aligned}\pi_i(q) = \underset{(p,s,u,o)\in\mathcal{A}(q)}{\arg\max} \Bigg\{ &\sum_{k\in\mathcal{S}} \frac{\Lambda_k(p)}{\Psi}\Big(p_k - (\gamma_k + r_k)\mathbf{1}_{\{o_k=1\}} - \upsilon_k(q_k^a/\lambda_k^\star - \delta_k)\mathbf{1}_{\{o_k=0\}} \\ &\qquad + (f_{i-1}(q^a+e_k) - f_{i-1}(q))\mathbf{1}_{\{o_k=0\}} + (f_{i-1}(q^a) - f_{i-1}(q))\mathbf{1}_{\{o_k=1\}}\Big) \\ &+ \sum_{k\in\mathcal{S}_{\mathrm{w}}} \frac{\ell_k[q_k^a]^+}{\Psi}\big(-(d_k+\gamma_k) + f_{i-1}(q^a - e_k) - f_{i-1}(q)\big) - \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \frac{h_k[q_k^a]^-}{\Psi} \\ &+ \frac{\mu_s}{\Psi}\big(-\gamma_s + f_{i-1}(q^a - e_s) - f_{i-1}(q)\big)\mathbf{1}_{\{s\neq 0\}}\Bigg\}, \qquad q\in\mathcal{Q}. \end{aligned} \tag{EC.110}$$

The following three lemmas characterize the production, rejection, and pricing decisions given the reallocation decisions. Then, using a simple search over the finite set of feasible reallocation decisions, we obtain the updated reallocation decisions and, consequently, the updated production, rejection, and pricing decisions.

LEMMA EC.23. *Given the reallocation decisions, the updated production decision at state $q \in \mathcal{Q}$ is given by*

$$s(\pi_i(q)) = \underset{\substack{s\in\{0\}\cup\{k\in\mathcal{S}^{\mathrm{MTO}}:q_k>0\}\\ \cup\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}:q_k>-M_k\}}}{\arg\max} \frac{\mu_s}{\Psi}\big(-\gamma_s + f_{i-1}(q^a - e_s) - f_{i-1}(q)\big)\mathbf{1}_{\{s\neq 0\}}.$$

Proof. The production decision s at state $q \in \mathcal{Q}$ only impacts the last term on the right-hand side of (EC.110). Therefore, given the reallocation decisions, the updated production decision can be characterized independently of the other decisions and is the maximizer of the last term on the right-hand side of (EC.110), which yields the result. □

LEMMA EC.24. *Given the reallocation decisions, the updated rejection decision for product $k \in \mathcal{S}$ at state $q \in \mathcal{Q}$ is given by*

$$o_k(\pi_i(q)) = \begin{cases} 0, & \text{if } f_{i-1}(q^a+e_k) - \upsilon_k(q_k^a/\lambda_k^\star - \delta_k) > f_{i-1}(q^a) - (\gamma_k + r_k) \text{ and } q_k < M_k, \\ 1, & \text{otherwise.}\end{cases}$$

Proof. The rejection decision o_k for product k only impacts the kth term on the right-hand side of (EC.110). Moreover, since $\Lambda_k(p) \geq 0$ for all $p \in \bar{\mathcal{P}}$ and $k \in \mathcal{S}$, the updated prices do not impact the updated rejection decisions. Therefore, the updated rejection decision o_k for product $k \in \mathcal{S}$ at state $q \in \mathcal{Q}$ satisfies

$$o_k(\pi_i(q)) = \operatorname*{arg\,max}_{o_k \in O_k(q)} \Big\{ -(\gamma_k + r_k)\mathbf{1}_{\{o_k=1\}} - \upsilon_k(q_k^a/\lambda_k^\star - \delta_k)\mathbf{1}_{\{o_k=0\}} \\ + (f_{i-1}(q^a + e_k) - f_{i-1}(q))\mathbf{1}_{\{o_k=0\}} + (f_{i-1}(q^a) - f_{i-1}(q))\mathbf{1}_{\{o_k=1\}} \Big\},$$

where $O_k(q) = \{1 - \mathbf{1}_{\{q_k < M_k\}}, 1\}$ denotes the set of feasible rejection decisions for product k at state q. The result follows by comparing the two expressions corresponding to $o_k = 0$ and $o_k = 1$, and selecting the value that yields the larger objective. □

LEMMA EC.25. *Given the reallocation decisions, the updated pricing decisions at state $q \in \mathcal{Q}$ are given by*

$$p(\pi_i(q)) = \operatorname*{arg\,max}_{p \in \bar{\mathcal{P}}} \sum_{k \in \mathcal{S}} \Lambda_k(p)\big(p_k - z_k(q)\big),$$

where, for $k \in \mathcal{S}$ and $q \in \mathcal{Q}$,

$$z_k(q) \coloneqq \big(\gamma_k + r_k - f_{i-1}(q^a)\big)\mathbf{1}_{\{o_k(\pi_i(q))=1\}} + \big(\upsilon_k(q_k^a/\lambda_k^\star - \delta_k) - f_{i-1}(q^a + e_k)\big)\mathbf{1}_{\{o_k(\pi_i(q))=0\}} + f_{i-1}(q).$$

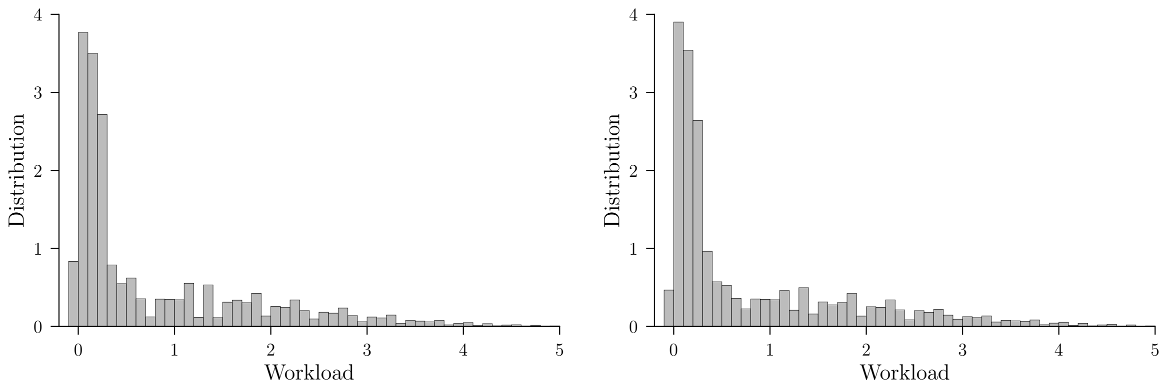
Proof. The pricing decision p at state $q \in \mathcal{Q}$ only impacts the first K terms on the right-hand side of (EC.110). Therefore, given the updated reallocation and rejection decisions, it follows that the updated pricing decision is characterized by the optimization problem in the statement of the lemma. □

Following the approach outlined in Gallego and Wang (2014, Section 2), we reformulate the optimization problem in Lemma EC.25 as a one-dimensional convex optimization problem. We then solve the reformulated problem using the Brent algorithm (Press 2007).

In summary, for each feasible reallocation decision at each state, we obtain the updated production, rejection, and pricing decisions as discussed above. Then, we iterate over the feasible reallocation decisions to find the updated reallocation decisions and, consequently, the updated policy; see (EC.110).

EC.5.3. Nested Logit Demand Model

In the numerical study in Section 8, we assume customers arrive to the system according to a Poisson process with rate Λ_0 and, upon arrival, choose a product according to a nested logit demand model; see Gallego and Topaloglu (2019). In this model, a customer first selects a nest of products and then chooses a specific product within that nest. This structure is natural in an omnichannel setting, where customers often make a coarse channel choice first, whether to purchase online or in store, and then choose among the products available within that channel. In our setting, there are two nests corresponding to the ordering



(a) Proposed policy with dynamic prices

(b) SMDP policy with dynamic prices

Figure EC.1 Distribution of workload under the proposed policy and the SMDP policy with dynamic prices.

Notes. Under both the proposed policy and the SMDP policy with dynamic prices, the workload process resides in the interval $[-0.06, 4.85]$ for over 99.9% of the simulation horizon. The workload distributions are plotted using bins of width 0.1.

channels, namely, online and walk-in. Let $\mathcal{N}_i$ denote the set of products in nest $i \in \{\text{Online}, \text{Walk-in}\}$, i.e., $\mathcal{N}_{\text{Online}} = \{2, 4\}$ and $\mathcal{N}_{\text{Walk-in}} = \{1, 3\}$. The probability that a customer selects nest i given the price vector p is denoted by $\mathbb{P}(\text{nest}\, i \,|\, p)$. Conditional on selecting nest i and the price vector p, the probability that a customer chooses product $k \in \mathcal{N}_i$ is denoted by $\mathbb{P}(\text{product}\, k \,|\, \text{nest}\, i,\, p)$. These probabilities are given as follows: For $i \in \{\text{Online}, \text{Walk-in}\}$ and $k \in \mathcal{N}_i$,

$$\mathbb{P}(\text{nest}\, i \,|\, p) = \frac{e^{c_i V_i}}{1 + \sum_{i \in \{\text{Online}, \text{Walk-in}\}} e^{c_i V_i}} \quad \text{and} \quad \mathbb{P}(\text{product}\, k \,|\, \text{nest}\, i,\, p) = \frac{e^{a_k - b_k p_k}}{\sum_{l \in \mathcal{N}_i} e^{a_l - b_l p_l}},$$

where $a_k \in \mathbb{R}$ captures the inherent value of product k, $b_k \in [0, \infty)$ captures the price sensitivity of product k, $V_i := \log(\sum_{l \in \mathcal{N}_i} e^{a_l - b_l p_l})$ represents the attractiveness of nest i, and $c_i \in (0, 1]$ captures the degree of product substitution within nest i. By the law of total probability, the probability that a customer chooses product $k \in \mathcal{N}_i$ given the price vector p is given by

$$\mathbb{P}(\text{product}\, k \,|\, p) = \mathbb{P}(\text{nest}\, i \,|\, p)\, \mathbb{P}(\text{product}\, k \,|\, \text{nest}\, i,\, p).$$

The nested logit demand function is then given by

$$\Lambda_k(p) = \Lambda_0\, \mathbb{P}(\text{product}\, k \,|\, p), \quad k \in \mathcal{S}, \quad p \in \mathbb{R}_+^K.$$

EC.5.4. Supplementary Numerical Results and Tables

This section presents supplementary results for the numerical study described in Section 8. Table EC.1 reports the mean and 95% interval for the prices used by each policy in the base case. Tables EC.2–EC.7 report the long-run average cost of the various policies and their gap from the SMDP benchmark (SMDP policy with dynamic prices). The reported values are sample averages; the standard errors for the long-run average costs and the gaps from the SMDP benchmark are less than or equal to 0.01 and 1.5%, respectively. Specifically, we conduct a sensitivity analysis in which one parameter is varied at a time, while all others are held fixed at their base case values. The procedure by which each parameter is varied is described in Section 8.2.

Table EC.1 Comparative evaluation of the prices used in the various policies under the base case.

	Product Class			
	Walk-in MTO	Walk-in MTS	Online MTO	Online MTS
SMDP policy with dynamic prices	1.026 (1.000, 1.096)	1.010 (1.000, 1.039)	1.027 (1.005, 1.108)	1.009 (1.000, 1.039)
Proposed policy with dynamic prices	1.025 (1.000, 1.101)	1.011 (1.000, 1.042)	1.025 (1.000, 1.101)	1.011 (1.000, 1.042)
Proposed policy with online-only dynamic prices	1.033	1.013	1.029 (1.000, 1.133)	1.013 (1.000, 1.056)
SMDP policy with static prices	1.047	1.019	1.047	1.019
Proposed policy with static prices	1.049	1.019	1.049	1.019
SMDP policy with nominal prices	1.000	1.000	1.000	1.000

Notes. The reported values are means, with 95% confidence intervals shown in parentheses.

Table EC.2 Comparative evaluation of long-run average costs across policies, varying only the fraction of (nominal) demand that chooses the MTS good.

	Nominal MTS Demand								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Long-run average expected cost									
SMDP policy with dynamic prices	0.38	0.36	0.35	0.33	0.32	0.30	0.27	0.25	0.22
Proposed policy with dynamic prices	0.38	0.37	0.35	0.34	0.32	0.30	0.28	0.26	0.23
Proposed policy with online-only dynamic prices	0.43	0.42	0.41	0.39	0.37	0.35	0.33	0.29	0.26
SMDP policy with static prices	0.52	0.51	0.48	0.47	0.44	0.42	0.39	0.36	0.32
Proposed policy with static prices	0.52	0.51	0.49	0.47	0.45	0.42	0.39	0.36	0.32
SMDP policy with nominal prices	1.76	1.69	1.63	1.57	1.51	1.21	1.04	0.97	0.87
Gap from the SMDP benchmark									
Proposed policy with dynamic prices	0.2%	0.9%	1.5%	0.6%	1.0%	2.6%	2.0%	1.8%	2.5%
Proposed policy with online-only dynamic prices	15.4%	14.2%	16.2%	15.5%	15.5%	18.5%	20.5%	16.8%	17.4%
SMDP policy with static prices	38.5%	39.4%	38.6%	39.3%	39.6%	41.2%	43.7%	44.5%	46.3%
Proposed policy with static prices	38.9%	41.2%	40.9%	39.9%	41.8%	41.9%	44.2%	44.1%	46.8%
SMDP policy with nominal prices	365%	365%	366%	369%	376%	383%	385%	389%	398%

Table EC.3 Comparative evaluation of long-run average costs across policies, varying only the fraction of (nominal) demand that chooses the online channel.

	Nominal Online Demand								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Long-run average expected cost									
SMDP policy with dynamic prices	0.51	0.44	0.39	0.35	0.32	0.30	0.29	0.27	0.26
Proposed policy with dynamic prices	0.51	0.44	0.39	0.35	0.32	0.31	0.29	0.28	0.27
Proposed policy with online-only dynamic prices	0.62	0.53	0.46	0.41	0.37	0.34	0.32	0.29	0.27
SMDP policy with static prices	0.63	0.56	0.51	0.47	0.44	0.42	0.40	0.38	0.36
Proposed policy with static prices	0.64	0.57	0.51	0.47	0.45	0.43	0.41	0.38	0.36
SMDP policy with nominal prices	1.91	1.78	1.69	1.59	1.51	1.44	1.39	1.31	1.28
Gap from the SMDP benchmark									
Proposed policy with dynamic prices	0.8%	0.2%	0.5%	1.1%	1.0%	1.5%	1.6%	2.0%	1.9%
Proposed policy with online-only dynamic prices	22.9%	21.2%	19.4%	17.6%	15.5%	13.1%	9.8%	5.9%	2.6%
SMDP policy with static prices	25.0%	27.9%	31.5%	35.3%	39.6%	40.1%	40.1%	38.9%	37.3%
Proposed policy with static prices	26.6%	29.0%	32.4%	36.3%	41.8%	41.3%	41.1%	39.2%	37.7%
SMDP policy with nominal prices	276%	304%	335%	358%	375%	372%	378%	379%	384%

Table EC.4 Comparative evaluation of long-run average costs across policies, varying only the abandonment rate.

	Abandonment Rate				
	0.0	0.5	1.0	1.5	2.0
Long-run average expected cost					
SMDP policy with dynamic prices	0.32	0.54	0.72	0.89	1.06
Proposed policy with dynamic prices	0.32	0.55	0.74	0.92	1.09
Proposed policy with online-only dynamic prices	0.37	0.58	0.77	0.94	1.11
SMDP policy with static prices	0.44	0.64	0.82	0.98	1.14
Proposed policy with static prices	0.45	0.65	0.82	0.99	1.15
SMDP policy with nominal prices	1.51	1.63	1.73	1.83	1.94
Gap from the SMDP benchmark					
Proposed policy with dynamic prices	1.0%	1.8%	2.6%	2.9%	3.7%
Proposed policy with online-only dynamic prices	15.5%	7.9%	6.7%	5.8%	5.5%
SMDP policy with static prices	39.6%	19.4%	14.4%	10.4%	7.8%
Proposed policy with static prices	41.8%	19.9%	14.2%	11.0%	9.4%
SMDP policy with nominal prices	377%	202%	140%	105%	83%

Table EC.5 Comparative evaluation of long-run average costs across policies, varying only the load factor.

	Load Factor							
	0.90	0.95	0.97	1.0	1.03	1.05	1.07	1.1
Long-run average expected cost								
SMDP policy with dynamic prices	0.14	0.20	0.24	0.32	0.44	0.56	0.70	0.95
Proposed policy with dynamic prices	0.14	0.20	0.24	0.32	0.45	0.57	0.71	0.97
Proposed policy with online-only dynamic prices	0.16	0.23	0.27	0.37	0.52	0.65	0.81	1.10
SMDP policy with static prices	0.16	0.23	0.31	0.44	0.64	0.80	0.99	1.34
Proposed policy with static prices	0.16	0.24	0.31	0.45	0.65	0.81	1.00	1.36
SMDP policy with nominal prices	0.16	0.33	0.58	1.51	3.07	4.42	5.89	8.22
Gap from the SMDP benchmark								
Proposed policy with dynamic prices	1.3%	1.1%	1.1%	1.0%	1.7%	1.9%	1.8%	1.8%
Proposed policy with online-only dynamic prices	14.3%	12.7%	13.4%	15.5%	17.9%	17.5%	16.0%	15.3%
SMDP policy with static prices	9.8%	16.8%	29.2%	39.6%	44.2%	44.2%	42.7%	40.5%
Proposed policy with static prices	15.2%	20.6%	30.2%	41.8%	45.7%	44.7%	43.7%	42.9%
SMDP policy with nominal prices	10%	66%	146%	376%	593%	693%	744%	765%

Table EC.6 Comparative evaluation of long-run average costs across policies, varying only the magnitude of the state (earliness, tardiness, and holding) costs.

	Magnitude of the State Costs							
	0.01	0.03	0.05	0.07	0.1	0.12	0.15	0.2
Long-run average expected cost								
SMDP policy with dynamic prices	0.12	0.23	0.32	0.40	0.51	0.59	0.69	0.88
Proposed policy with dynamic prices	0.12	0.23	0.32	0.41	0.53	0.60	0.71	0.90
Proposed policy with online-only dynamic prices	0.13	0.26	0.37	0.47	0.59	0.69	0.81	1.01
SMDP policy with static prices	0.17	0.32	0.44	0.55	0.69	0.78	0.89	1.10
Proposed policy with static prices	0.17	0.33	0.45	0.55	0.70	0.78	0.90	1.12
SMDP policy with nominal prices	0.78	1.21	1.51	1.74	2.02	2.18	2.39	2.69
Gap from the SMDP benchmark								
Proposed policy with dynamic prices	1.2%	1.7%	1.0%	1.0%	3.1%	2.0%	2.3%	2.2%
Proposed policy with online-only dynamic prices	12.8%	13.5%	15.5%	15.9%	15.6%	17.7%	17.1%	14.8%
SMDP policy with static prices	41.5%	40.3%	39.6%	36.7%	34.2%	32.4%	28.8%	25.9%
Proposed policy with static prices	47.5%	43.5%	41.8%	37.6%	35.5%	33.4%	29.3%	27.4%
SMDP policy with nominal prices	550%	430%	377%	336%	293%	271%	244%	207%

Table EC.7 Comparative evaluation of long-run average costs across policies, varying only the correlation parameter of the nested logit demand model.

	Nested Logit Correlation Parameter									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Long-run average expected cost										
SMDP policy with dynamic prices	0.26	0.28	0.29	0.30	0.30	0.31	0.31	0.31	0.32	0.32
Proposed policy with dynamic prices	0.27	0.28	0.30	0.31	0.31	0.31	0.32	0.32	0.32	0.32
Proposed policy with online-only dynamic prices	0.28	0.32	0.34	0.34	0.35	0.36	0.36	0.37	0.37	0.37
SMDP policy with static prices	0.32	0.37	0.40	0.41	0.42	0.42	0.43	0.43	0.44	0.44
Proposed policy with static prices	0.32	0.37	0.40	0.41	0.42	0.43	0.43	0.44	0.44	0.45
Gap from the SMDP benchmark										
Proposed policy with dynamic prices	1.5%	1.5%	2.0%	2.9%	2.1%	1.3%	1.5%	1.6%	0.7%	1.0%
Proposed policy with online-only dynamic prices	7.9%	14.1%	15.4%	14.2%	17.2%	16.7%	16.5%	17.9%	15.4%	15.5%
SMDP policy with static prices	21.1%	31.7%	35.8%	35.9%	37.6%	37.3%	38.9%	39.3%	38.1%	39.6%
Proposed policy with static prices	21.4%	32.1%	36.6%	36.7%	38.0%	39.6%	39.7%	39.9%	39.1%	41.8%